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On the Scattering and Absorption of Light in Gaseous Media, with Applications to the Intensity of Sky Radiation

Louis Vessot King

Phil. Trans. R. Soc. Lond. A 1913 **212**, 375-433
doi: 10.1098/rsta.1913.0012

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XII. *On the Scattering and Absorption of Light in Gaseous Media, with Applications to the Intensity of Sky Radiation.*

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Communicated by Sir JOSEPH LARMOR, *Sec. R.S.*

Received June 7,—Read November 7, 1912.

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PART I.

§ 1. *On the Scattering of Parallel Radiation by Molecules and Small Particles.*

THE effect of small particles in scattering incident radiation was first worked out by Lord RAYLEIGH.* When a stream of parallel radiation falls on a particle whose dimensions are small compared with the wave-length the resulting secondary disturbance travels in all directions at the expense of the intensity in the original direction. In a later paper Lord RAYLEIGH† gave reasons for believing that the molecules of a gas are themselves able to scatter radiation in this way. In a gaseous medium it is legitimate to sum up the intensities of the scattered radiation due to each molecule in an element of volume without a consideration of phase-difference in consequence of the continuous change in the relative positions of a molecule in a gas.

* RAYLEIGH, 'Phil. Mag.,' XLI., pp. 107, 274, 447 (1871); XII., p. 81 (1881); 'Collected Works,' vol. I., pp. 87, 104, 518.

† RAYLEIGH, 'Phil. Mag.,' XLVII., pp. 375–384 (1899); 'Collected Works,' vol. IV., pp. 397–405.

 Published separately, February 8, 1913.

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The same remark applies to the case where the scattering is due to small particles of dust since these partake, to some extent at least, of the molecular agitation of the gas in which they are held in suspension.

The intensity of radiation at any point is measured by the amount of energy crossing unit area of a surface normal to the direction of the radiation in unit time. Unpolarized radiation of intensity E falls on an element δv of a gas of density ρ and containing N molecules per unit volume. The intensity of the scattered radiation in a direction θ with the incident radiation and at a distance r from δv we denote by $r^{-2}\mathrm{I}(r, \theta)\,\delta v$, so that $\mathrm{I}(r, \theta)\,\delta v\,\delta\omega$ is the energy contained in a small solid angle $\delta\omega$ crossing a spherical surface at distance r in unit time.

The expression for $\mathrm{I}(0, \theta)$ is of the form

$$\mathrm{I}(0, \theta) = \mu(\theta)\,\mathrm{E} \quad \ldots \ldots \ldots \quad (1)$$

where $\mu(\theta)$ depends on the direction θ and is proportional to the number N of molecules per unit volume, *i.e.*, $\mu(\theta)$ is also proportional to the density ρ.

If $\mu_0(\theta)$, N_0, ρ_0 refer to values of $\mu(\theta)$, N, ρ under determinate conditions of pressure and temperature we have

$$\mu(\theta)/\mu_0(\theta) = \mathrm{N}/\mathrm{N}_0 = \rho/\rho_0 \quad \ldots \ldots \ldots \quad (2)$$

$\mu(\theta)$ may be expressed in terms of the optical properties of the medium: the results of RAYLEIGH,* and KELVIN† worked out on various hypotheses of the molecule and of the æther agree in giving rise to the expression,

$$\mu(\theta) = \tfrac{1}{2}\pi^2(n^2-1)^2\lambda^{-4}(1+\cos^2\theta)/\mathrm{N} \quad \ldots \ldots \ldots \quad (3)$$

where n is the refractive index of the gas and λ the wave-length of the incident radiation.

SCHUSTER‡ has recently obtained this result from general considerations independently of any particular theory.

Since $n-1$ and N are both proportional to the density of the gas we notice that $\mu(\theta)$ is also proportional to the density, as already stated.

Formula (3) shows that the intensity of the scattered radiation is twice as great in the direction of the incident radiation as it is in a direction at right angles. Considerable simplification in the analysis is obtained by employing the mean value of $\mu(\theta)$ over a spherical surface. Thus, writing

$$4\pi\bar{\mu} = \int \mu(\theta)\,d\omega \quad \ldots \ldots \ldots \quad (4)$$

(3) gives

$$\bar{\mu} = \tfrac{2}{3}\pi^2(n^2-1)^2\lambda^{-4}/\mathrm{N} \quad \ldots \ldots \ldots \quad (5)$$

* RAYLEIGH, 'Phil. Mag.,' XLI., p. 107; 'Collected Works,' vol. I., p. 87.

† KELVIN, 'Baltimore Lectures' (1904), p. 311.

‡ SCHUSTER, 'Theory of Optics,' 2nd ed. (1909), p. 325.

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and under standard conditions of pressure and temperature,

$$\bar{\mu}_0 = \tfrac{2}{3}\pi^2 (n_0^2-1)^2 \lambda^{-4}/N_0 \quad . \quad . \quad . \quad (6)$$

We denote by κ_0 the expression

$$\kappa_0 = 4\pi\bar{\mu}_0 = \tfrac{8}{3}\pi^3 (n_0^2-1)^2 \lambda^{-4}/N_0 . \quad . \quad . \quad (7)$$

The effect of scattering is to diminish the intensity of the incident radiation, giving rise to the phenomenon of attenuation especially noticeable in the diminution of intensity of solar radiation in its passage through the earth's atmosphere. The consideration of attenuation as due to scattering alone involves the assumption that energy is nowhere accumulating in the gas. In order to give greater generality to the application of the analysis we introduce a term expressing the fact that the temperature at any point is increasing.

If E be the intensity of radiation crossing unit area of a plane at a point x in unit time, the loss to E in a distance dx in unit time due to the conversion of radiant energy into molecular agitation represented by a rise of temperature is of the form

$$dE = -\alpha E\, dx . \quad . \quad . \quad . \quad (8)$$

when α is proportional to the number of molecules per unit volume, *i.e.*, if α_0 and ρ_0 refer to standard conditions of pressure and temperature, $\alpha/\alpha_0 = \rho/\rho_0$.

In the case of a pure gas α is a quantity depending on the distribution of energy between vibrating systems within the molecules and the motions of the molecules themselves which define the temperature of a gas on the kinetic theory scale as proportional to the mean squares of molecular velocities.

If ρ be the density of the gas, s its specific heat at constant pressure and $d\Theta$ the increment of temperature in time dt due to the conversion of radiant energy $\alpha E\; dx\, dt$ into heat, we have

$$\alpha E = \rho s \frac{d\Theta}{dt} , \quad . \quad . \quad . \quad (9)$$

E being measured in calories per unit area normal to the direction of E per second.

In the case of a gas exposed continuously to external radiation we may suppose the temperature to attain to a steady state when $d\Theta/dt = 0$ and therefore $\alpha = 0$ in which case no energy accumulates in the medium. In the problem of the earth's atmosphere, however, the term α will give rise to a small diurnal variation of temperature throughout the atmosphere. The term α may also be taken to include the effect of dust in absorbing solar radiation without scattering as well as effects of selective absorption if we regard α as a function of the wave-length. The existence and magnitude of this quantity can be determined by a comparison of theoretical results with the results of observation. Actual numerical values for air are obtained in Part III. of the present paper.

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§ 2. *On the General Integral Equation for the Scattered Radiation.*

Suppose a mass of gas to be completely enclosed by a boundary Σ and illuminated by a distribution of radiation whose intensity at a point (x, y, z) is E, and whose direction is defined at that point. Each element of volume will scatter a certain proportion of the radiation incident upon it, so that each element besides being illuminated by the incident radiation is also subject to the aggregate radiation from all the other elements within the surface Σ, *i.e.*, to the effect of *self-illumination.* This constitutes what SCHUSTER* has called the problem of "Radiation through a Foggy Atmosphere." The problem does not lend itself easily to a complete formulation in terms of differential equations, but can be expressed in terms of an *integral equation.*

At an element of volume δv at (x, y, z) the incident radiation is E. At a distance r from δv in a direction θ with the incident radiation the scattered radiation crossing unit area in unit time is denoted by $\mathrm{I}(r, \theta)\,\delta v\, r^{-2}$. If we wish to express the dependence of $\mathrm{I}(r, \theta)\,\delta v\, r^{-2}$ on the position of δv in the surface Σ we write it in the form $\mathrm{I}(x, y, z, r, \theta)\,\delta v\, r^{-2}$.

Our first problem will be to express the scattered radiation $\mathrm{I}(r, \theta)$ in terms of the scattered radiation $\mathrm{I}(0, \theta)$ per unit solid angle in the neighbourhood δv.

Consider the radiation $\mathrm{I}(r, \theta)\,\delta v\,\delta\omega$ coming from an element of volume δv and contained in a small solid angle $\delta\omega$ in a direction θ with the incident radiation. The intensity of the radiation crossing a spherical surface of radius r is $\mathrm{I}(r, \theta)\,\delta v\,\delta\omega$; that cutting the surface $r+\delta r$ is

$$\left\{\mathrm{I}(r, \theta)+\frac{\delta}{\delta r}\mathrm{I}(r, \theta)\,\delta r\right\}\delta v\,\delta\omega.$$

We thus obtain the equation

$$-\frac{\delta}{\delta r}\mathrm{I}(r, \theta)\,\delta v\,\delta\omega = \alpha\mathrm{I}(r, \theta)\,r^{-2}\,\delta v\, r^2\,\delta\omega\,\delta r + 4\pi\bar{\mu}\mathrm{I}(r, \theta)\,r^{-2}\,\delta v\, r^2\,\delta\omega\,\delta r. \quad . \quad . \quad (10)$$

The first term on the right-hand side represents by (8) the energy lost to the element of volume $r^2\,\delta\omega\,\delta r$ by the conversion of radiant energy into a rise of temperature or into long-wave heat radiation, which transformation we assume to go on at a constant rate.

The second term accounts for the energy lost to the element of volume $r^2\,\delta\omega\,\delta r$ by scattering. We neglect the effect of self-illumination within the small solid angle $\delta\omega$ since this only enters into the equation to the second order of small quantities.

* SCHUSTER, "Radiation through a Foggy Atmosphere," 'Astrophysical Journal,' XXI, January, 1905, p. 1. A somewhat similar problem had been previously considered by the same writer in a paper "The Influence of Radiation on the Transmission of Heat," 'Phil. Mag.,' February, 1903. Recently it has been shown by JACKSON, W. H. ('Bull. Am. Soc. Math.,' XVI, June, 1910, p. 473), that the generalized differential equation obtained in SCHUSTER's paper can be transformed into an *integral equation* of the Fredholm type.

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Writing $\kappa = 4\pi\bar{\mu}$ the above equation gives

$$-\frac{\delta}{\delta r}\mathrm{I}(r, \theta) = (\alpha+\kappa)\,\mathrm{I}(r, \theta). \qquad (11)$$

If we write

$$\mathrm{K} = \alpha+\kappa, \qquad (12)$$

the solution of (11) is

$$\mathrm{I}(r, \theta) = \mathrm{I}(0, \theta)\, e^{-\int_0^r \mathrm{K}\, dr}, \qquad (13)$$

which puts into evidence the variation of α and κ with the density of the gas.

Consider now the radiation incident on an element of volume δv at (x, y, z):—

(i) The external illumination E which contributes to the scattered radiation from δv an amount

$$\mu(\theta)\,\mathrm{E}(x, y, z)\,\delta v\,\delta\omega;$$

(ii) An element of volume $\delta v'$ at (x', y', z') gives rise to an intensity $\mathrm{I}(x', y', z', r', \theta')\,\delta v'\, r'^{-2}$ at the point (x, y, z): this contributes to the scattered radiation from δv the amount

$$\mu(\widehat{rr'})\,\mathrm{I}(x', y', z', r', \theta')\,\delta v'\,\delta\omega\,\delta v\, r'^{-2},$$

$\widehat{rr'}$ denoting the angle between r and r'. The total contribution to the scattered radiation from δv due to self-illumination by the entire volume Σ is

$$\delta\omega\,\delta v \int_\Sigma \mu(\widehat{rr'})\,\mathrm{I}(x', y', z', r', \theta')\, r'^{-2}\, dv',$$

the integral being taken throughout the entire volume enclosed by the surface Σ.

By definition the sum of contributions (i) and (ii) is $\mathrm{I}(x, y, z, 0, \theta)$, so that we obtain the following *integral equation* for the scattered radiation at and from any point,

$$\mathrm{I}(x, y, z, 0, \theta) = \mu(\theta)\,\mathrm{E}(x, y, z) + \int_\Sigma \mu(\widehat{rr'})\,\mathrm{I}(x', y', z', 0, \theta')\, r'^{-2}\, e^{-\int_0^{r'} \mathrm{K}\, dr'}\, dv', \qquad (14)$$

$\mathrm{I}(x', y', z', r', \theta')$ being expressed in terms of $\mathrm{I}(x', y', z', 0, \theta')$ by means of (13).

A differential equation involving $\mathrm{E}(x, y, z)$ is obtained by considering the rate of accumulation of energy in an element of volume.

As soon as $\mathrm{I}(x, y, z, 0, \theta)$ is known as a function of the position of the point O, (x, y, z), in the volume bounded by the surface Σ, the radiation scattered to any point P within the boundary Σ contained in a small solid angle ω is given by the formula

$$\mathrm{T}\omega = \omega \int_0^{r_0} \mathrm{I}(x, y, z, 0, \theta)\, e^{-\int_0^r \mathrm{K}\, dr}\, dr, \qquad (15)$$

where $r = \mathrm{PO}$, $r_0 = \mathrm{PQ}$, the radius vector to the boundary, and T is the intensity at

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P for unit solid angle. The formula (15) is easily modified to include the case where P is without the surface Σ.

It is interesting to point out the analogy which the method of the present section bears to the ordinary procedure of potential theory. The function $I(x, y, z)$ corresponds to a potential function and is expressed in terms of an external effect $E(x, y, z)$ by means of an *integral equation.* The *total intensity* in a small solid angle, the effect which is physically measurable, is derived by *integrating* between definite limits along a given direction.

Further progress towards a solution of (14) is impossible without a number of simplifying assumptions which are best considered in dealing with a particular problem, such, for instance, as that presented by the effect of the earth's atmosphere in absorbing and scattering the sun's radiation.

Part II.

§ 3. *Application to Radiation and Absorption in the Earth's Atmosphere.*

In the following sections we shall assume as an approximation that the surface of the earth is a plane and that the density is a function of the height above the earth's surface only. We also neglect effects due to reflection from the earth's surface and to refraction by the earth's atmosphere.

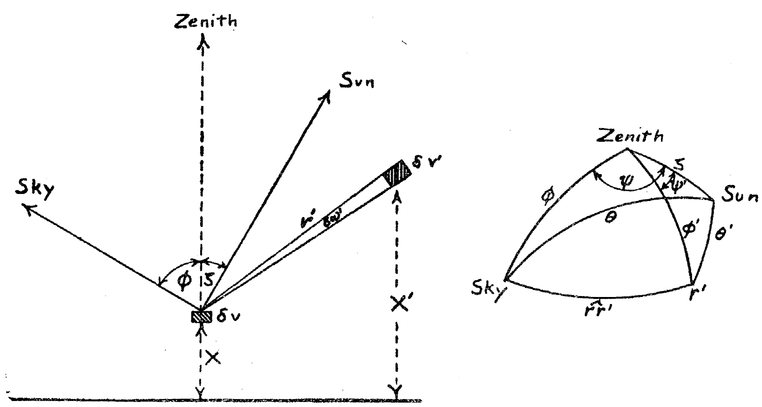

Fig. 1.

The integral equation (14) for the scattered radiation in a direction θ (fig. 1) with the direction of the sun and coming from an element δv of the atmosphere at a height x above the earth's surface, takes the form

$$I(x, 0, \theta) = \mu(\theta) E(x) + \int_{\Sigma} \mu(\widehat{rr'}) \frac{I(x', 0, \theta')}{r'^2} e^{-\int_0^{r'} K\, dr'} dv, \quad \ldots \ldots (16)$$

where the integral is taken throughout the entire atmosphere.

In order to simplify the solution of (16) we assume that the scattered radiation

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from a molecule can to a first approximation be taken equal in all directions, *i.e.*, the equation can be reduced to one with a single variable by writing

$$\mu(\theta) = \mu(\widehat{rr'}) = \bar{\mu}.$$

Equation (16) may now be written, since $dv' = r'^2\,d\omega'\,dr'$,

$$\mathrm{I}(x) = \bar{\mu}\mathrm{E}(x) + \bar{\mu}\int_{\Sigma} \mathrm{I}(x')\,e^{-\int_0^{r'}\mathrm{K}\,dr'}\,d\omega'\,dr'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Remembering that $\bar{\mu}/\bar{\mu}_0 = \kappa/\kappa_0 = \alpha/\alpha_0 = \mathrm{K}/\mathrm{K}_0 = \rho/\rho_0$, we now employ the transformations

$$\mathrm{R} = \int_0^r \frac{\rho}{\rho_0}\,dr, \quad \mathrm{X} = \int_0^x \frac{\rho}{\rho_0}\,dx, \quad \mathrm{H} = \int_0^\infty \frac{\rho}{\rho_0}\,dx, \quad \mathrm{J}(\mathrm{X}) = \frac{\rho_0}{\rho}\mathrm{I}(x), \quad . \quad . \quad (18)$$

and (17) now takes the form

$$\mathrm{J}(\mathrm{X}) = \bar{\mu}_0\mathrm{E}(\mathrm{X}) + \bar{\mu}_0\int \mathrm{J}(\mathrm{X}')\,e^{-\mathrm{K}_0\mathrm{R}'}\,d\omega'\,d\mathrm{R}', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

while (15) transforms into

$$\mathrm{T}\omega = \omega\int_0^{\mathrm{R}_0} \mathrm{J}(\mathrm{X})\,e^{-\mathrm{K}_0\mathrm{R}}\,d\mathrm{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The integral on the right-hand side of (19) must now be taken throughout a *homogeneous* atmosphere of density ρ_0 included between the planes $\mathrm{X} = 0$ and $\mathrm{X} = \mathrm{H}$.

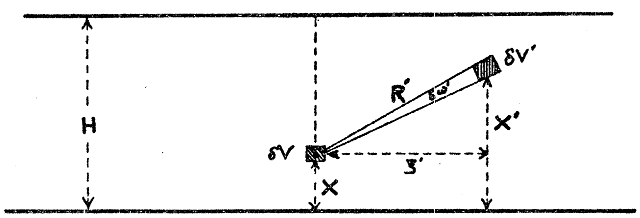

Fig. 2.

Expressing the integral in (18) in cylindrical co-ordinates (ξ', ψ') (fig. 2), we have

$$d\omega'\,d\mathrm{R}' = \frac{d\mathrm{V}'}{\mathrm{R}'^2} = \frac{d\mathrm{X}'\,\xi'\,d\xi'\,d\psi'}{\xi'^2 + (\mathrm{X}'-\mathrm{X})^2},$$

where ψ' is the azimuth of the element of volume $d\mathrm{V}'$ referred to some fixed direction of reference. The integral then becomes

$$\int_{\mathrm{X}'} \mathrm{J}(\mathrm{X}')\,d\mathrm{X}'\int_0^{2\pi} d\psi'\int_0^\infty e^{-\mathrm{K}_0[\xi'^2+(\mathrm{X}'-\mathrm{X})^2]^{1/2}}\frac{\xi'\,d\xi'}{\xi'^2+(\mathrm{X}'-\mathrm{X})^2}.$$

The last integral when integrated with respect to ξ' gives

$$\int_0^\infty e^{-\mathrm{K}_0[\xi'^2+(\mathrm{X}'-\mathrm{X})^2]^{1/2}}\frac{\xi'\,d\xi'}{\xi'^2+(\mathrm{X}'-\mathrm{X})^2} = -Ei\{-\mathrm{K}_0(\mathrm{X}'-\mathrm{X})\} \quad \text{when} \quad \mathrm{X}<\mathrm{X}'<\mathrm{H}$$

$$\text{and} \quad = -Ei\{-\mathrm{K}_0(\mathrm{X}-\mathrm{X}')\} \quad \text{when} \quad 0<\mathrm{X}'<\mathrm{X},$$

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where $Ei(-x)$ is GLAISHER'S* *Exponential Integral* denoted by

$$Ei(-x) = -\int_x^\infty u^{-1}e^{-u}\,du. \quad \ldots \quad (21)$$

The integral equation (19) now takes the form

$$J(X) = \bar{\mu}_0 E(X) - 2\pi\bar{\mu}_0\left[\int_0^X J(X')Ei\{-K_0(X-X')\}dX' + \int_X^H J(X')Ei\{-K_0(X'-X)\}dX'\right]. \quad (22)$$

[The differential equation by means of which E may be expressed as a function of X can be obtained by a consideration of the rate of accumulation of energy of the direct solar radiation between the planes x and $x+dx$. An analysis similar to that by means of which equation (13) was obtained, together with the transformations of equation (18), lead to the expression

$$E(X) = Se^{K_0(X-H)\sec\zeta}, \quad \ldots \quad (23)$$

where S is the intensity of solar radiation outside the earth's atmosphere corresponding to a given wave-length and ζ is the zenith distance of the sun.

If we consider the rate of accumulation of the total energy (including both direct and scattered radiation) between the planes x and $x+dx$, it can be shown by making use of the integral equation connecting the direct with the scattered radiation that the exponential law of transmission expressed in (23) is valid.†]

It is well to state clearly the assumptions involved in obtaining (13), (19), and (23):—

(i) In obtaining the differential equation leading to (13) the direct radiation is considered independently of the scattered radiation within the small solid angle ω.

(ii) The integral equation (22) assumes as an approximation that the radiation scattered by an element of volume is distributed equally in all directions.

(iii) With these two conditions it can be shown from a consideration of attenuation of total radiation in a thin layer dx that the ordinary exponential law of transmission (23) follows; *i.e.*, that the transmission of direct radiation may be considered independently of the scattered radiation.

(iv) By means of the transformation (18) it is shown that the problem of scattered

* GLAISHER, 'Phil. Trans.,' 1870, p. 367.

[† *Note added September 20, 1912.*—The calculation referred to was given at length in the paper as originally communicated; the analysis is, however, somewhat tedious and hardly necessary in view of the fact that (13) is obtained according to the assumption that the direct radiation is considered independently of the scattered radiation within the small solid angle ω. It is to be expected, therefore, that the attenuation of direct radiation in a parallel beam of solar radiation may be considered independently of the scattered radiation consistently with the above assumption leading immediately to (23); the consideration of the attenuation of the total radiation confirms this point and leads to the same result. The writer is indebted to the referees for the above suggestion.]

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radiation in the earth's atmosphere reduces to the case of an atmosphere of uniform density contained between two parallel planes X = 0 and X = H. This transformation is independent of any law of density with height, provided the planes of equal density are parallel to the earth's surface.

§ 4. *On the Approximate Solution of Integral Equations.*

The integral equation (21) is of the Fredholm type,*

$$u(x) = f(x) + \int_{x_1}^{x_2} u(\xi) \, \mathrm{K}(x, \xi) \, d\xi. \quad \ldots \ldots \ldots \quad (24)$$

Except for special forms of the *kernel* $\mathrm{K}(x, \xi)$ the formal solutions of (24) do not lend themselves easily to numerical evaluation: we therefore develop a method of approximation which applies with sufficient accuracy to the problem in hand.

Suppose for all values of x between x_1 and x_2 that $f(x)$ lies between A and a ($\mathrm{A} > a$). Then to a first approximation

$$u(x) \text{ lies between } \mathrm{A} + \mathrm{A}\int_{x_1}^{x_2} \mathrm{K}(x, \xi) \, d\xi \quad \text{and} \quad a + a\int_{x_1}^{x_2} \mathrm{K}(x, \xi) \, d\xi,$$

provided $\mathrm{K}(x, \xi)$ is everywhere positive.

We write

$$\phi(x) = \int_{x_1}^{x_2} \mathrm{K}(x, \xi) \, d\xi, \quad \ldots \ldots \ldots \quad (25)$$

then if for all values of x, $\phi(x)$ lies between B and b ($\mathrm{B} > b$) we have to a second approximation

$$a + (a + ab) \, b < u(x) < \mathrm{A} + (\mathrm{A} + \mathrm{AB}) \, \mathrm{B},$$

or

$$a(1 + b + b^2) < u(x) < \mathrm{A}(1 + \mathrm{B} + \mathrm{B}^2).$$

A repetition of the process shows that

$$a(1 + b + b^2 + b^3 + \ldots) < u(x) < \mathrm{A}(1 + \mathrm{B} + \mathrm{B}^2 + \mathrm{B}^3 + \ldots).$$

If $|\mathrm{B}| < 1$ both series are convergent and

$$\epsilon_1 < u(x) < \epsilon_2 \quad \text{where} \quad \epsilon_1 = a/1 - b \quad \text{and} \quad \epsilon_2 = \mathrm{A}/1 - \mathrm{B}. \quad \ldots \quad (26)$$

Substituting in (24) we see that the solution $u(x)$ lies between the limits

$$u_1(x) = f(x) + \epsilon_1 \phi(x), \quad \text{and} \quad u_2(x) = f(x) + \epsilon_2 \phi(x). \quad \ldots \quad (27)$$

$u_1(x)$ and $u_2(x)$ may be called the *extreme solutions.*

* *Cf.* Bôcher, M., 'An Introduction to the Study of Integral Equations,' Cambridge, 1909, p. 14; also Bateman, H., "Report on the History and Present State of the Theory of Integral Equations," 'Brit. Assoc. Report,' 1910, p. 25.

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If now α represent the mean value of $f(x)$ between x_1 and x_2, *i.e.*,

$$\alpha = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and β stand for the mean value of $\phi(x)$ between x_1 and x_2, *i.e.*,

$$\beta = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \phi(x)\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

while $\epsilon = \alpha/(1-\beta)$, the solution $\bar{u}(x) = f(x) + \epsilon\phi(x)$ may be called the *mean solution*, while the three solutions may be expressed in a single formula by the notation

$$u(x) = f(x) + \begin{pmatrix}\epsilon_1 \\ \epsilon \\ \epsilon_2\end{pmatrix}\phi(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

We notice that $u_1(x) < u(x) < u_2(x)$. In the applications to be considered, $u_1(x)$ and $u_2(x)$ are sufficiently close to warrant the use of the mean solution $\bar{u}(x)$. As long as $\bar{u}(x)$ does not depart far from the arithmetical mean of the extreme solutions, $\frac{1}{2}\{u_1(x) + u_2(x)\}$, we may take the value of $\bar{u}(x)$ to represent an approximation not far removed from the exact solution of the integral equation (24).

It is perhaps worth noting here that to a higher degree of approximation the approximate solution of the integral equation may be written

$$u(x) = f(x) + \int_{x_1}^{x_2} f(\xi)\,\mathrm{K}(x,\xi)\,d\xi + \begin{pmatrix} b\epsilon_1 \\ \beta\epsilon \\ \mathrm{B}\epsilon_2\end{pmatrix}\phi(x). \quad . \quad . \quad . \quad . \quad . \quad (31)$$

In the following sections the solutions (31) involve the evaluation of troublesome integrals, so that the simpler but somewhat less accurate approximations given in (30) will be employed.

§ 5. *On the Solution of the Integral Equation for Sky Radiation.*

In the integral equation (22) we may express to a first approximation the dependence of the scattered radiation in any direction on the angle which that direction makes with the incident radiation by retaining the term $\mu_0(\theta)$ instead of $\bar{\mu}_0$ in the first term of the right-hand side of the equation which may be written, on making use of (23),

$$\mathrm{J}(\mathrm{X}) = \mu_0(\theta)\,\mathrm{S}e^{-\mathrm{K}_0(\mathrm{H}-\mathrm{X})\sec\zeta} - \tfrac{1}{2}\kappa_0\left[\int_0^{\mathrm{X}} \mathrm{J}(\mathrm{X}')\,Ei\{-\mathrm{K}_0(\mathrm{X}-\mathrm{X}')\}\,d\mathrm{X}' + \int_{\mathrm{X}}^{\mathrm{H}} \mathrm{J}(\mathrm{X}')\,Ei\{-\mathrm{K}_0(\mathrm{X}'-\mathrm{X})\}\,d\mathrm{X}'\right]. \quad . \quad (32)$$

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In the notation of the preceding section we have to solve the above integral equation for

$$u(\mathrm{X}) = \mathrm{J}(\mathrm{X})/\mathrm{S}\mu_0(\theta).$$

We have

$$\mathrm{A} = 1 \quad \text{and} \quad a = e^{-\mathrm{K}_0\mathrm{H}\sec\zeta}, \qquad (33)$$

while

$$\phi(\mathrm{X}) = -\tfrac{1}{2}\kappa_0\left[\int_0^{\mathrm{X}} Ei\{-\mathrm{K}_0(\mathrm{X}-\mathrm{X}')\}\,d\mathrm{X}' + \int_{\mathrm{X}}^{\mathrm{H}} Ei\{-\mathrm{K}_0(\mathrm{X}'-\mathrm{X})\}\,d\mathrm{X}'\right].$$

If we notice that

$$\int_c^{\infty} -Ei(-ax)\,dx = f(ac)/a, \qquad (34)$$

where

$$f(x)^* = e^{-x} + xEi(-x) = x\int_x^{\infty} e^{-u}u^{-2}\,du, \qquad (35)$$

we find

$$\phi(\mathrm{X}) = \tfrac{1}{2}[2 - f(\mathrm{K}_0\mathrm{X}) - f\{\mathrm{K}_0(\mathrm{H}-\mathrm{X})\}]\,\kappa_0/\mathrm{K}_0. \qquad (36)$$

This expression is symmetrical with respect to the plane $\mathrm{X} = \frac{1}{2}\mathrm{H}$, where it has its maximum value. The minimum value of the expression occurs at $\mathrm{X} = 0$ and $\mathrm{X} = \mathrm{H}$.

We write for brevity

$$\mathrm{C} = \mathrm{K}_0\mathrm{H}, \qquad c = \kappa_0\mathrm{H}, \qquad \gamma = \alpha_0\mathrm{H}, \qquad \mathrm{C} = c + \gamma. \qquad (37)$$

We then find

$$\mathrm{B} = \{1 - f(\tfrac{1}{2}\mathrm{C})\}\,c/\mathrm{C} \quad \text{and} \quad b = \tfrac{1}{2}\{1 - f(\mathrm{C})\}\,c/\mathrm{C}. \qquad (38)$$

We also have

$$\alpha = \mathrm{H}^{-1}\int_0^{\mathrm{H}} e^{-\mathrm{K}_0(\mathrm{H}-\mathrm{X})\sec\zeta}\,d\mathrm{X} = (1 - e^{-\mathrm{C}\sec\zeta})/\mathrm{C}\sec\zeta, \qquad (39)$$

or, introducing the notation

$$\mathrm{G}(x) = (1 - e^{-x})/x, \qquad (40)$$

we have

$$\alpha = \mathrm{G}(\mathrm{C}\sec\zeta). \qquad (41)$$

Further, we find

$$\beta = \mathrm{H}^{-1}\int_0^{\mathrm{H}} \phi(\mathrm{X})\,d\mathrm{X},$$

or

$$\beta = \tfrac{1}{2}\mathrm{H}^{-1}\left[\int_0^{\mathrm{H}} \{1 - f(\mathrm{K}_0\mathrm{X})\}\,d\mathrm{X} + \int_0^{\mathrm{H}} [1 - f\{\mathrm{K}_0(\mathrm{H}-\mathrm{X})\}]\,d\mathrm{X}\right] c/\mathrm{C},$$

$$\beta = \mathrm{C}^{-1}\left[\int_0^{\mathrm{C}} \{1 - f(u)\}\,du\right] c/\mathrm{C}.$$

* The function $f(x)$, as defined above, occurs in a number of absorption problems. Its properties are described in a paper by the writer (KING, L. V., 'Phil. Mag.,' February, 1912, p. 245), where a short numerical table of the function is given.

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It can easily be shown by integrating by parts that

$$\int_0^C f(u)\,du = \tfrac{1}{2}(1-e^{-C}) + \tfrac{1}{2}C f(C), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

so that

$$\beta = \{1-\tfrac{1}{2}f(C) - \tfrac{1}{2}G(C)\}\,c/C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Finally we obtain for ϵ_2, ϵ, ϵ_1 the expressions

$$\left.\begin{aligned} \epsilon_2 &= \frac{1}{\gamma/C + f(\frac{1}{2}C)\,c/C} \\ \epsilon &= \frac{G(C\sec\zeta)}{\gamma/C + \frac{1}{2}\{f(C)+G(C)\}\,c/C}, \\ \epsilon_1 &= \frac{e^{-C\sec\zeta}}{\gamma/C + \frac{1}{2}\{1+f(C)\}\,c/C}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The approximate solution of (32) may now be written

$$\frac{J(X)}{\mu_0(\theta)\,S} = e^{-K_0(H-X)\sec\zeta} + \begin{pmatrix}\epsilon_1\\ \epsilon\\ \epsilon_2\end{pmatrix}\phi(X), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where $\phi(X)$ is defined by (36) and the values ϵ_1, ϵ, ϵ_2 are given by (44) and are employed in (45) according as we wish to make use of the extreme or mean solutions.

From (20) the radiation scattered to a point on the earth's surface, contained in a small solid angle ω in a direction ϕ with the vertical, is given by

$$\omega T = \omega \int_0^{R_0} J(X)\,e^{-K_0 R}\,dR.$$

Since $R = X\sec\phi$ this equation becomes

$$T(\phi, \zeta) = \sec\phi \int_0^H e^{-K_0 X\sec\phi}\,J(X)\,dX, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

where the dependence of the sky radiation per unit solid angle on the direction ϕ and on the zenith distance of the sun ζ is denoted by $T(\phi, \zeta)$.

If we denote by $\omega R(\phi, \zeta)$ the radiation to a point at $X = H$ contained in a small solid angle ω pointing earthwards in a direction ϕ with the vertical, we have

$$R(\phi, \zeta) = \sec\phi \int_0^H e^{-K_0(H-X)\sec\phi}\,J(X)\,dX. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Substituting for $J(X)$ from (45) we obtain from (46)

$$T(\phi, \zeta) = \mu_0(\theta)\,S\sec\phi\left\{e^{-K_0 H\sec\zeta}\int_0^H e^{-K_0 X(\sec\phi-\sec\zeta)}\,dX + \begin{pmatrix}\epsilon_1\\ \epsilon\\ \epsilon_2\end{pmatrix}\int_0^H e^{-K_0 X\sec\phi}\,\phi(X)\,dX\right\}.$$

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The first term in the brackets may be written

$$\left.\begin{array}{l} e^{-\mathrm{C}\sec\phi}\,\mathrm{G}\left\{(\sec\zeta-\sec\phi)\,\mathrm{C}\right\}\mathrm{C}/\mathrm{K}_0 \text{ if } \zeta > \phi \\ \text{or} \\ e^{-\mathrm{C}\sec\zeta}\,\mathrm{G}\left\{(\sec\phi-\sec\zeta)\,\mathrm{C}\right\}\mathrm{C}/\mathrm{K}_0 \text{ if } \phi > \zeta \end{array}\right\} \quad (48)$$

The evaluation of the second integral is more difficult; we have

$$\int_0^{\mathrm{H}} e^{-\mathrm{K}_0\mathrm{X}\sec\phi}\phi(\mathrm{X})\,d\mathrm{X} = \tfrac{1}{2}\frac{c}{\mathrm{C}}\int_0^{\mathrm{H}} e^{-\mathrm{K}_0\mathrm{X}\sec\phi}\left[2-f(\mathrm{K}_0\mathrm{X})-f\{\mathrm{K}_0(\mathrm{H}-\mathrm{X})\}\right]d\mathrm{X}. \quad (49)$$

Writing $\mathrm{K}_0\mathrm{X} = u$, the above integral takes the form

$$\frac{1}{2\mathrm{K}_0}\frac{c}{\mathrm{C}}\left[\frac{2\,(1-e^{-\mathrm{C}\sec\phi})}{\sec\phi} - \int_0^{\mathrm{C}} e^{-u\sec\phi}f(u)\,du - e^{-\mathrm{C}\sec\phi}\int_0^{\mathrm{C}} e^{u\sec\phi}f(u)\,du\right]. \quad (50)$$

We denote by $\mathrm{B}(x)$ and $\mathrm{B}(-x)$ the functions

$$\mathrm{B}(x) = Ei(-x)-\log x, \qquad \mathrm{B}(-x) = Ei(x)-\log x. \quad (51)$$

The expansion when x is small of the exponential integral is

$$Ei(x) = \gamma+\tfrac{1}{4}\log x^4+x+\frac{x^2}{2\,.\,2\,!}+\frac{x^3}{3\,.\,3\,!}+\frac{x^4}{4\,.\,4\,!}+\ldots, \quad (52)$$

where γ is Euler's constant, $\gamma = 5572$ and the expansion holds for both positive and negative values of x.

Thus when x is small the expansions for $\mathrm{B}(x)$ and $\mathrm{B}(-x)$ are

$$\left.\begin{array}{l} \mathrm{B}(x) = \gamma-x+\dfrac{x^2}{2\,.\,2\,!}-\dfrac{x^3}{3\,.\,3\,!}+\ldots \\ \text{and} \\ \mathrm{B}(-x) = \gamma+x+\dfrac{x^2}{2\,.\,2\,!}+\dfrac{x^3}{3\,.\,3\,!}+\ldots \end{array}\right\} \quad (53)$$

It can be shown that the integral

$$\int_0^c f(ax)\,e^{-bx}\,dx = \frac{1}{b}\{1-e^{-bc}f(ac)\} + \frac{a}{b}\left[\frac{1}{b}\log\left|\frac{a}{a+b}\right| - \frac{1}{b}e^{-bc}Ei(-ac)+\frac{1}{b}Ei\{-(a+b)\,c\}\right]. \quad (54)$$

The result holds for positive as well as negative values of b provided the argument of $(a+b)\,c$ in the logarithm and in $Ei\{-(a+b)\,c\}$ is the same.

By means of this result and in the notation defined in (51) it can be shown that (50) reduces to

$$\tfrac{1}{2}\frac{c}{\mathrm{C}}\frac{1}{\mathrm{K}_0}\frac{1}{\sec\phi}\Phi(\mathrm{C},\phi) \quad (55)$$

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where

$$\Phi(C,\phi) = (1-e^{-C\sec\phi})\{1-f(C)\}$$
$$+\cos\phi[B(C)-B\{C(1+\sec\phi)\}+e^{-C\sec\phi}\{B(C)-B[-C(\sec\phi-1)]\}]. \quad (56)$$

We notice that for $\phi = 0$, $\sec\phi = 1$,

$$\Phi(C,0) = (1-e^{-C})\{1-f(C)\}+B(C)-B(2C)+e^{-C}\{B(C)-\gamma\}, \quad (57)$$

where γ here stands for EULER'S constant.

For $\phi = \frac{1}{2}\pi$, $\sec\phi = \infty$ (56) reduces to

$$\Phi(C,\tfrac{1}{2}\pi) = 1-f(C). \quad (58)$$

From (48) and (55) the expression for the intensity at the earth's surface of radiation from that portion of sky which is in any direction ϕ with the vertical and azimuth ψ measured from a vertical plane through the sun is, for $\zeta > \phi$.

$$T(\phi,\zeta) = \frac{\mu_0(\theta)}{K_0} S\left[C\sec\phi e^{-C\sec\phi} G\{C(\sec\zeta-\sec\phi)\}+\frac{1}{2}\frac{c}{C}\begin{pmatrix}\epsilon_1\\ \epsilon\\ \epsilon_2\end{pmatrix}\Phi(C,\phi)\right], \quad (59)$$

while for $\zeta < \phi$ we have

$$T(\phi,\zeta) = \frac{\mu_0(\theta)}{K_0} S\left[C\sec\phi e^{-C\sec\zeta} G\{C(\sec\phi-\sec\zeta)\}+\frac{1}{2}\frac{c}{C}\begin{pmatrix}\epsilon_1\\ \epsilon\\ \epsilon_2\end{pmatrix}\Phi(C,\phi)\right]. \quad (60)$$

From equations (3) and (7) we notice that

$$\mu_0(\theta) = \tfrac{3}{4}(1+\cos^2\theta)\bar{\mu}_0 = \tfrac{3}{4}(1+\cos^2\theta)\kappa_0/4\pi,$$

so that the factor

$$S\mu_0(\theta)/K_0 = S(4\pi)^{-1}\tfrac{3}{4}(1+\cos^2\theta)c/C. \quad (61)$$

From the polar diagram fig. 1 we see that

$$\cos\theta = \cos\phi\cos\zeta+\sin\phi\sin\zeta\cos\psi. \quad (62)$$

The first terms in equations (59) and (60) give the contribution to the intensity of sky light due to the sun's radiation which has been once scattered by the atmosphere.

If C is small and if the attenuation is due to scattering only, ($c = C$), the first term of (60) gives for the scattered radiation coming from a direction ϕ the value

$$(4\pi)^{-1}S\tfrac{3}{4}(1+\cos^2\theta)c\sec\phi \quad (63)$$

which agrees with the value obtained by KELVIN.*

The first terms in (59) and (60) taken as they stand with $C = c$ take into account the fact that both the incident and scattered radiations suffer attenuation: this effect

* KELVIN, *loc. cit.* equation (19), p. 313.

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is considered in a formula obtained by RAYLEIGH.* The term γ in $C = c+\gamma$ allows for attenuation by absorption alone (*i.e.*, without scattering). The second terms in (59) and (60) represent the contribution of self-illumination to the scattered radiation coming from any particular direction. An evaluation of this effect has not, so far as the writer is aware, been submitted to calculation although the importance of the effect is realized both by KELVIN† and RAYLEIGH,* and in an analogous problem by LOMMEL.‡

The expressions (59) and (60) for the scattered radiation from any direction besides depending on the coefficients of absorption for the radiation of wave-length under consideration, depend also on the angular co-ordinates of direction ϕ and ψ as well as on ζ the zenith distance of the sun. If we consider the intensity from zenith sky the expressions are greatly simplified. Writing $\phi = 0$ and $\theta = \zeta$, (59) gives,

$$T(0, \zeta) = \frac{\mu_0(\zeta)}{K_0} S \left[Ce^{-C} G\{C(\sec\zeta - 1)\} + \tfrac{1}{2}\frac{c}{C}\begin{pmatrix}\epsilon_1\\ \epsilon\\ \epsilon_2\end{pmatrix}\Phi(C, 0)\right], \quad . \quad . \quad . \quad (64)$$

where $\Phi(C, 0)$ is given by (57) and is tabulated in Table V.

It is not difficult to construct a double-entry table in terms of C and ζ giving the values for the functions which occur in (64) so that observations on zenith sky are most appropriate for comparison with the results of calculation from the attenuation coefficients determined by observations at the *same* time.

The intensity of sky radiation from the direction of the horizon $\left(\phi = \frac{\pi}{2}\right)$ is given by

$$T\left(\frac{\pi}{2}, \zeta\right) = \frac{\mu_0(\theta) S}{K_0}\left\{ e^{-C\sec\zeta} + \tfrac{1}{2}\frac{c}{C}\begin{pmatrix}\epsilon_1\\ \epsilon\\ \epsilon_2\end{pmatrix}\Phi(C, \tfrac{1}{2}\pi)\right\}, \quad . \quad . \quad . \quad . \quad (65)$$

where by (62) θ is given by $\cos\theta = \sin\zeta\cos\phi$ and $\Phi(C, \frac{1}{2}\pi)$ is defined by (58). It will be noticed that this formula unlike the approximate one (63), obtained by KELVIN,

* RAYLEIGH, 'Phil. Mag.,' XLI., p. 116.

† KELVIN, *loc. cit.*, p. 302, sect. 54.

‡ LOMMEL, E., 'Sitzungb. der math.-phys. Classe der K. Bayer. Akad. der Wiss.,' Bd. 17 (1887), p. 95; (the analysis is reproduced by MÜLLER, 'Photometrie der Gestirne,' Leipzig, 1897, pp. 47–52). In working out the scattering of radiation by opaque, diffusely reflecting surfaces on an absorption theory, LOMMEL obtains a first order approximation to the effect of self-illumination in giving rise to deviations from LAMBERT'S Law of diffuse reflection, the modified formula being referred to as the Lommel-Seeliger Law of Illumination. This problem is included as a particular case of the investigation of the present paper. LOMMEL'S formula for diffuse reflection is represented in the present instance by equation (69), which is the solution of an integral equation expressing exactly the effect of self-illumination. MÜLLER points out that even LOMMEL'S modification of LAMBERT'S Law does not represent exactly the results of observation on diffuse reflection; it may happen that the more complete solution represented by (69), adapted to the case of intense absorption, gives a better representation of fact. This point must, however, be left over for further investigation.

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remains finite, although its application to the case of the earth's atmosphere is somewhat invalidated by the curvature of the earth.

On examining (59) it will be noticed that as long as C is not too large we may write

$$T(\phi, \zeta) = \frac{\mu_0(\theta)}{\mu_0(\zeta)} \sec\phi T(0, \zeta) = \frac{1+\cos^2\theta}{1+\cos^2\zeta} \sec\phi T(0, \zeta), \quad . \quad . \quad . \quad . \quad (66)$$

which may be taken as a rough approximation for the intensity of sky-radiation from any direction not too close to the horizon where (65) must be used. When C is large the wave-length corresponding is small, and the intensity in the normal solar spectrum outside the atmosphere also becomes small. Hence the formula (66) is sufficiently accurate when the total intensity on a horizontal plane is required. Denoting by $H(\zeta)$ the intensity of scattered radiation of wave-length λ received per unit time on unit area of horizontal surface, we have

$$H(\zeta) = \int T(\phi, \zeta) \cos\phi \, d\omega,$$

the integral being taken over the entire sky.

Since $d\omega = \sin\phi \, d\phi \, d\psi$, we have

$$H(\zeta) = \int_0^{2\pi} d\psi \int_0^{1/2\pi} T(\phi, \zeta) \cos\phi \sin\phi \, d\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

and making use of the approximation in (66) we have

$$H(\zeta) = \frac{T(0, \zeta)}{1+\cos^2\zeta} \int_0^{2\pi} d\psi \int_0^{1/2\pi} (1+\cos^2\theta) \sin\phi \, d\phi.$$

The total intensity for all wave-lengths per unit area of a horizontal surface is given by

$$\int_0^\infty H(\zeta) \, d\lambda = \frac{2\pi}{\frac{3}{4}(1+\cos^2\zeta)} \int_0^\infty T(0, \zeta) \, d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Of some interest is the intensity of the radiation which is scattered from the atmosphere back into interplanetary space. If we write (H−X) for X in (47) we notice that

$$R(\phi, \zeta) = \sec\phi \int_0^H e^{-K_0 X \sec\phi} J(H-X) \, dX;$$

we also notice from (36) that $\phi(H-X) = \phi(X)$.

We thus obtain from (45)

$$R(\phi, \zeta) = \mu_0(\theta) S \sec\phi \left[\int_0^H e^{-K_0 X (\sec\phi + \sec\zeta)} dX + \begin{pmatrix} \epsilon_1 \\ \epsilon \\ \epsilon_2 \end{pmatrix} \int_0^H e^{-K_0 X \sec\phi} \phi(X) \, dX \right],$$

which reduces to

$$R(\phi, \zeta) = \frac{\mu_0(\theta) S}{K_0} \left[C \sec\phi G \{C(\sec\phi + \sec\zeta)\} + \tfrac{1}{2} \frac{c}{C} \begin{pmatrix} \epsilon_1 \\ \epsilon \\ \epsilon_2 \end{pmatrix} \Phi(C, \phi) \right]. \quad . \quad . \quad (69)$$

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We notice that $R(\phi, \zeta)$ is only approximately equal to $T(\phi, \zeta)$ when C is small, *i.e.*, that the usual assumption that as much radiation is scattered in the direction of incidence as in the opposite direction only holds approximately when the coefficient of attenuation is small.

§ 6. *Note on the Polarization of Sky Radiation.*

It is well known that sky radiation is partly polarized in a vertical plane passing through the position of the sun (the principal plane): in so far as the radiation to be scattered is direct solar radiation, the polarization ought to be complete. That portion of the sky radiation due to self-illumination is largely unpolarized and may to a large extent account for this deficiency from complete polarization: this point is mentioned by RAYLEIGH in his 1871 paper and the analysis of the present paper enables the magnitude of this factor to be roughly estimated. The complete solution of the problem from this aspect would require us to split up the incident radiation into two components, one of which is polarized in the principal plane, the other at right angles to it: the effect of self-illumination would lead to two simultaneous integral equations in three variables, the solution of which would be much too complicated to be useful.

If, however, we refer to equations (59) and (60) it will be noticed that the expression for the intensity of sky radiation may be written in the form

$$T(\phi, \zeta) = \mu_0(\theta) S \{P(\phi, \zeta) + Q(\phi, \zeta)\}/K_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where $P(\phi, \zeta)$ stands for the first term in the brackets of (59) or (60) and

$$Q(\phi, \zeta) = \tfrac{1}{2} \frac{c}{C} \begin{pmatrix} \epsilon_1 \\ \epsilon \\ \epsilon_2 \end{pmatrix} \Phi(C, \phi)$$

represents the effect of self-illumination.

In default of a rigorous solution it is not unreasonable to suppose that the portion of the scattered radiation due to self-illumination is independent of the angle of polarization of the incident radiation. As far as the primary scattered radiation alone is concerned, the intensities polarized in the principal plane and in a plane at right angles to it are in the proportion 1 to $\cos^2\theta$. Thus from (70) the ratio of the sky intensities polarized in the principal plane and in the plane at right angles to it are given by the ratio

$$\frac{T_1(\phi, \zeta)}{T_2(\phi, \zeta)} = \frac{P(\phi, \zeta) + Q(\phi, \zeta)}{\cos^2\theta P(\phi, \zeta) + Q(\phi, \zeta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

If we make use of the approximations of equation (66),

$$P(\phi, \zeta) = \sec\phi \, P(0, \zeta) \, \mu_0(\theta)/\mu_0(\zeta), \qquad Q(\phi, \zeta) = \sec\phi \, Q(0, \zeta) \, \mu_0(\theta)/\mu_0(\zeta)$$

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(71) may be written

$$\frac{T_2(\phi, \zeta)}{T_1(\phi, \zeta)} = \frac{\cos^2\theta + Q(0, \zeta)/P(0, \zeta)}{1 + Q(0, \zeta)/P(0, \zeta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

From this formula we see that sky polarization is most complete for $\theta = \frac{1}{2}\pi$, *i.e.*, over a great circle an angular distance $\frac{1}{2}\pi$ from the sun polarization would be complete in light from this direction if $Q(0, \zeta)$ were zero, *i.e.*, if the effects of self-illumination were negligible.

For the sake of comparison with observation the ratio $T_1(\phi, \zeta)/T_2(\phi, \zeta)$ is calculated for zenith sky, $(\phi = 0)$, in the form

$$\frac{T_1(0, \zeta)}{T_2(0, \zeta)} = \frac{1 + Q(0, \zeta)/P(0, \zeta)}{\cos^2\zeta + Q(0, \zeta)/P(0, \zeta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

Part III.

§7. *Analysis of Observations on the Attenuation of Solar Radiation by the Earth's Atmosphere.*

The intensity of solar radiation transmitted to a station at a height x above sea-level is given by equation (23) in the form

$$E(X) = Se^{-K_0(H-X)\sec\zeta},$$

where

$$X = \int_0^x \frac{\rho}{\rho_0}\,dx, \quad \text{and} \quad H = \int_0^\infty \frac{\rho}{\rho_0}\,dx$$

represents the height of the "homogeneous atmosphere."

Thus

$$H - X = \int_x^\infty \frac{\rho}{\rho_0}\,dx, \quad \text{so that} \quad \frac{H-X}{H} = \frac{\int_x^\infty \rho\,dx}{\int_0^\infty \rho\,dx} = \frac{p}{p_0},$$

where p and p_0 are the pressures of the atmosphere at the station and at the sea-level respectively. We thus have, writing $C_x = Cp/p_0$, the expression

$$E(x) = Se^{-C_x\sec\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

If the heights of the barometer at the station and at sea-level are known at the time of observation, the comparison of absorption coefficients is independent of the law of variation of atmospheric pressure-gradient with height.

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We notice from (37), writing $C = c+\gamma = \kappa_0 H + \alpha_0 H_0$, that

$$i.e.,\quad \left.\begin{aligned} C &= \tfrac{32}{3}\pi^3 (n_0-1)^2 \lambda^{-4} H/N_0 + \alpha_0 H, \\ {}^*C &= \beta\lambda^{-4}+\gamma, \quad \text{where} \quad \beta = \tfrac{32}{3}\pi^3 (n_0-1)^2 H/N_0, \end{aligned}\right\} \quad \ldots \quad (75)$$

which gives

$$C_x p_0/p = \beta\lambda^{-4}+\gamma. \quad \ldots \quad (76)$$

Extensive observations on the determination of the coefficient of attenuation C_x for different wave-lengths have been carried out at various stations by the work of the Smithsonian Astrophysical Observatory.† The mean coefficients of atmospheric transmission for Washington, Mount Wilson, and Mount Whitney have recently been given by ABBOT.‡ These are quoted in Table I., while for the sake of independent comparison, the results of MÜLLER§ for Potsdam are also added.

In order to study the correctness of formula (76) the coefficients of transmission for the different stations are plotted on a base λ^{-4}, λ being measured in microns (10^{-4} cm.). The results are shown in Diagram I., and give rise to a number of straight lines. If the absorption were due to air-molecules alone, (76) shows that we should obtain a family of straight lines all passing through the same point ($\lambda^{-4} = -\gamma/\beta$). The straight lines actually obtained show that some variable factor in the atmosphere other than the molecules themselves is effective in attenuation, especially for stations below the level of Mount Wilson. This factor is generally referred to as atmospheric "dust." A slight generalization of the analysis by which (76) was obtained enables us to interpret the results shown graphically in Diagram I.

Let N' be the number of "dust" particles per unit volume at a height x above the earth's surface. The coefficient of attenuation, which includes the effects of scattering and absorption both by air molecules and by "dust," may be written in the form

$$C_x = (\beta\lambda^{-4}+\gamma)\,p/p_0 + K'_0(\lambda)\int_x^\infty \frac{N'}{N'_0}\,dx, \quad \ldots \quad (77)$$

where N'_0 is a constant representing the number of dust particles per unit volume at the earth's surface, and $K'_0(\lambda)$ depends on the nature of absorption and scattering by the dust particles. The distribution of "dust" N' in the atmosphere may be regarded

* The representation of the coefficient of attenuation as the sum of two terms, one constant and the other varying inversely as the fourth power of the wave-length, seems to have been first recognized by BECKER (see KELVIN, 'Baltimore Lectures,' p. 321, equation 33) from an analysis of MÜLLER'S observations.

† 'Annals of the Smithsonian Astrophysical Observatory,' vol. II., by C. G. ABBOT and F. E. FOWLE, Washington, 1908. Referred to subsequently as 'Annals,' vol. II.

‡ ABBOT, C. G., "The Sun's Energy-Spectrum and Temperature," 'Astrophysical Journal,' XXXIV., October, 1911.

§ MÜLLER, G., "Photometrie der Gestirne," Leipzig, 1897, p. 140.

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as an unknown and variable quantity. Diagram I. shows that for Mount Wilson and for Mount Whitney the effect of "dust" is small, *i.e.*, we may take $N' = 0$ for x greater than 1780 metres. In this case (77) reduces to (76) and the attenuation may be taken to be due almost entirely to the effect of air-molecules, while the existence of a small term, γ, indicates that even in the comparatively dust-free air above Mount Whitney there is a small amount of attenuation by absorption, *i.e.*, a direct conversion of solar radiation into thermal agitation of atmospheric molecules.

We notice from Table II. that for the comparatively dust-free air above the level of Mount Wilson the value of γ under standard conditions of pressure is $\gamma = \alpha_0 H = \cdot 032$, or, since $H = 7\cdot 988 \times 10^5$ cm. at 0° C., the value of α_0 is

$$\alpha_0 = 4\cdot 0 \times 10^{-8} \text{ cm.}^{-1}. \qquad (78)$$

On referring to (8) we notice that α_0 is the fraction of radiant energy converted per centimetre of path into thermal molecular agitation. This fraction is greatly increased by the presence of small solid particles such as "dust," &c. From (9) we can estimate the rate of increase of temperature in a gas under atmospheric pressure due to solar radiation passing through it.

We have $\dfrac{d\Theta}{dt} = \dfrac{\alpha_0 E}{\rho_0 s}$ from equation (9).

Taking the value of α_0 for dust-free air from (78) and writing $E = 1\cdot 92$ calories per minute, $\rho_0 = \cdot 001293$, $s = \cdot 237$, we find

$$\left.\begin{aligned}\frac{d\Theta}{dt} &= 2\cdot 5 \times 10^{-4} \text{ degrees C. per minute} \\ &= 1\cdot 5 \times 10^{-2} \text{ degrees C. per hour}\end{aligned}\right\} \qquad (79)$$

In the case of ordinary air at sea-level, the Washington observations from Table II. show that the rate of increase of temperature calculated in (79) must be increased to about six times this value.

With regard to sea-level stations, Diagram I. seems to indicate, both for the Washington and Potsdam observations, a marked change in the nature of the absorption due to "dust" in the neighbourhood of $\cdot 610\mu$. We therefore discuss separately the case of long-wave radiation ($\lambda > \cdot 610\mu$) and short-wave radiation ($\lambda < \cdot 610\mu$).

(i) *Long-wave Radiation* ($\lambda > \cdot 610\mu$).

For long waves the straight lines of Diagram I. show that the term $K'_0(\lambda)$ of (77) must be of the form

$$K'_0(\lambda) = \beta''\lambda^{-4} + \gamma'',$$

where β'' and γ'' are constants for the range of wave-lengths greater than $\cdot 610\mu$. This

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indicates that the presence of "dust" gives rise to both absorption and scattering; (77) may then be written

$$C_x = \frac{1}{\lambda^4}\left\{\frac{p}{p_0}\beta + \beta''\int_0^\infty \frac{N'}{N'_0}\,dx\right\} + \Gamma''\frac{p}{p_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

where

$$\Gamma'' = \gamma + \frac{p_0}{p}\gamma''\int_x^\infty \frac{N'}{N'_0}\,dx.$$

Diagram I. shows clearly that for long waves the straight lines both for Potsdam and Washington intersect in the same point, $\lambda^{-4} = -\gamma/\beta$, as do the lines for Mount Wilson and Mount Whitney.

From (80) this result requires that

$$\beta/\gamma = \beta''/\gamma'', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

a condition which is independent of the law of distribution of the "dust" particles.

The ratio $\frac{\beta}{\gamma}$ is proportional to the ratio

$$\frac{\text{energy of incident wave scattered by small particles}}{\text{energy of incident wave converted into molecular agitation}},$$

and (81) indicates from the results of observation that for long-wave radiation this ratio is independent of the nature of the scattering particles, whether "dust" or air-molecules. This result throws some light on the question raised at the end of §1 as to the mechanism by which the molecules of a gas can convert a portion of the radiant energy incident upon them into thermal molecular agitation. The same mechanism which is effective in scattering radiation is also capable of effecting molecular velocities, and therefore the rate of increase of temperature in such a way that the ratio β/γ is independent of the nature of the molecule or even of the small "dust" particle giving rise to the absorption and scattering.

(ii) *Short-wave Radiation* ($\lambda < \cdot 610\mu$).

For short waves we may suppose that the incident radiation is not scattered by the dust-particles but is absorbed and converted into heat. On this supposition $K'_0(\lambda)$ in (77) is of the form γ', where γ' is a constant independent of the wave-length. Equation (77) then takes the form

$$C_x = \lambda^{-4}\beta p/p_0 + \gamma p/p_0 + \gamma'\int_x^\infty \frac{N'}{N'_0}\,dx = \lambda^{-4}\beta p/p_0 + \Gamma' p/p_0. \quad . \quad . \quad . \quad (82)$$

The slope of this line in the graphical representation of Diagram I. is given by

$$\tan\theta' = \beta p/p_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

and is independent of the distribution of "dust" in the atmosphere. This conclusion is justified by calculating β for the various stations from the slope of the lines in

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Diagram I. and a knowledge of the mean barometric pressures at these stations. The results are tabulated and described in Table II. From these values of β we may by (75) calculate N_0 the number of molecules in a gas under standard conditions of pressure and temperature. We take $n_0 - 1 = \cdot 000293$, $H = 7 \cdot 988 \times 10^5$ cm. at 0° C. and (71) then gives

$$N_0 = 2 \cdot 269 \times 10^{17}/\beta. \quad \ldots \quad (84)$$

The results for each set of observations are given in Table II. The values of N_0 agree remarkably well among themselves, and give for a mean value*

$$N_0 = 2 \cdot 32 \times 10^{19}. \quad \ldots \quad (85)$$

This result is in substantial agreement with the value of N_0 obtained by RUTHERFORD and GEIGER†

$$N_0 = 2 \cdot 72 \times 10^{19},$$

and with the value obtained by MILLIKAN‡ from a recent determination of the elementary electrical charge, $e = 4 \cdot 891 \times 10^{-10}$ E.S.U., which gives

$$N_0 = 2 \cdot 644 \times 10^{19},$$

taking the Faraday constant to be 9·655 absolute E.M. units. This agreement indicates that for short-wave radiation the scattering is almost entirely due to air-molecules, while the effect of "dust" is to produce a genuine absorption effect, *i.e.* a direct conversion of radiant energy into heat.

The term

$$\Gamma' = \gamma + \frac{p_0}{p} \gamma' \int_x^\infty \frac{N'}{N'_0} \, dx. \quad \ldots \quad (86)$$

involves the distribution of atmospheric "dust" and may therefore be expected to be an extremely variable factor with respect both to place and time. The method of analysis of the present section offers a convenient method of studying the variations in the distribution of atmospheric "dust" and their connection with other meteorological phenomena.

* The close agreement of coefficients of attenuation calculated from the formula $c = \beta/\lambda^{-4}$, using RUTHERFORD and GEIGER'S value of N_0, with coefficients calculated from observations on *selected clear days* at Washington and Mount Wilson, was pointed out by SCHUSTER ("Molecular Scattering and Atmospheric Absorption," 'Nature,' July 22, 1909; 'Optics,' 2nd ed., 1909, p. 329).

† RUTHERFORD, E., and GEIGER, H., "Charge and Nature of the α-Particle," 'Roy. Soc. Proc.,' A, vol. 81, 1908, p. 171.

‡ MILLIKAN, R. A., "The Isolation of an Ion, a Precision Measurement of its Charge, and the Correction of STOKES' Law," 'Phys. Rev.,' XXXII., April, 1911. A summary of the various physical measurements which lead to the value of N_0 is given by PERRIN, J. ('Annales de Chimie et de Physique,' 8me sér., September, 1909, translated by SODDY, F.; 'Brownian Movement and Molecular Reality,' Taylor and Francis, 1910, p. 90).

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§ 8. *On the Intensity of Sky Radiation as Calculated from the Mean Coefficients of Attenuation at Mount Wilson and Washington.*

It will be noticed from (64) that the intensity from zenith sky for different wave-lengths can be expressed in terms of the zenith distance of the sun and the coefficients of attenuation C, c, and γ which are determined for a given station by an analysis of atmospheric transmission observations according to the method of the preceding section. In order to simplify the calculations from (64) the various functions of C and ζ which occur in this formula are tabulated in Tables III., IV., and V.

The intensity from zenith sky is then worked out for the two stations Mount Wilson and Washington from the mean coefficients of attenuation at these two places. In the first case the numerical values corresponding to the extreme and mean solutions of the integral equation are carried throughout all the calculations and in this way give the limit of errors due to an approximate solution. In the second case, which is taken as typical of a sea-level station, numerical values corresponding to the mean solution of the integral equation are alone given, the reason being that the absorption at sea-level is an extremely variable quantity which would give rise to fluctuations in sky radiation probably exceeding the difference of the extreme solutions.

Zenith intensities of sky radiation for different wave-lengths and for various zenith distances of the sun are given in Table VI. for Mount Wilson and in Table VIII. for Washington. The unit of intensities is arbitrary and is that employed in Table I. for the normal solar spectrum outside the earth's atmosphere. The results are shown graphically in Diagrams II. to VII. for Mount Wilson and in Diagram XII. for Washington. From these curves it is possible to obtain by double interpolation the intensity from zenith sky corresponding to any wave-length and zenith distance of the sun. In this way a comparison was made of the quality of sky radiation obtained by calculation with that obtained experimentally at Mount Wilson.* The results are given numerically in Table X. and are compared graphically in Diagram XIII.

By integrating the curves for the zenith intensity and making use of the approximate formula (66) we are able to obtain a rough approximation to the total intensity of sky radiation from any direction. On comparing (66) with (65), for which $\phi = \frac{1}{2}\pi$, it must be noted that the first of these formulæ can only be used for zenith distances which are not too great. It will also be noticed that $T(\frac{1}{2}\pi, \zeta)$ does not vary as rapidly with the wave-length as $T(0, \zeta)$. The nature of the scattered radiation from a portion of the sky near the horizon is governed principally by the term $Se^{-C\sec\zeta}$, *i.e.*, it is of nearly the same quality as the direct sunlight except for the contribution to the intensity due to the second term in $T(\frac{1}{2}\pi, \zeta)$ which represents the effect of self-illumination. This feature is not shown by

* 'Annals,' vol. II., Table 32, p. 155.

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expressions for sky radiation hitherto obtained; in fact the whitish colour of the sky near the horizon was considered by KELVIN* to be an objection to the theory of scattering.

By calculating the values of the intensities of solar radiation which reaches the earth's surface for various zenith distances of the sun and for various wave-lengths, and by integrating the curves so obtained, we obtain the values of the total intensity of solar radiation reaching the two stations Mount Wilson and Washington. Taking the value of the solar constant to be 1·922 calories per square centimetre per minute,† the results are given in the same units and are calculated for solar radiation incident on a plane normal to the sun's rays as well as for the solar radiation incident on a horizontal plane. The results are given in Tables VII. and IX., and are shown graphically in Diagrams IX. and X.

These results enable the value of the total sky intensity in any direction to be compared with the intensity of direct solar radiation at the station in the form *sky/sun* for equal solid angles (semi-diameter of sun taken as 16′ of arc). The results are given in Tables VII. and IX., both for Mount Wilson and Washington, and are shown graphically in Diagram XI.

By interpolation from the diagram just mentioned the values of *sky/sun* for various directions of sky and for various zenith distances of the sun are compared with the values observed at Mount Wilson.‡ The comparison is made in Tables XI. and XIII.; the results of calculation are in fair agreement with observation except for regions of sky near the sun and near the horizon. The first of these discrepancies is probably brought about by the simplifying assumption made in equation (17) by writing $\bar{\mu}$ for $\mu(\theta)$ and $\mu(\widehat{rr'})$. The existence of such a term depending on an angular co-ordinate complicates the integral equation beyond hope of solution; it can be seen, however, that its existence gives rise to a bright region of sky in the neighbourhood of the sun not represented by the approximate solution considered in the present paper. The discrepancy which exists in the case of directions of sky near the horizon is due to the failure of the approximate formula (66) for large values of ϕ. In such cases the complete formula (60) should be employed. The use of this formula would require the tabulation of functions of two and three variables, which might be undertaken when more numerous and more accurate observations on sky radiation are available. It must be remembered in making these comparisons that the result of sky observations on certain specified days are compared with values based on mean coefficients of attenuation. The only satisfactory method of making the comparison is to obtain sky radiation observations on the same day as observations are made for the coefficients of attenuation of solar

* KELVIN, *loc. cit.*, p. 307.

† ABBOT, C. G., and FOWLE, F. E., "The Value of the Solar Constant of Radiation," 'Astrophysical Journal,' XXXIII., April, 1911, p. 191.

‡ 'Annals,' vol. II., Table 32, p. 151.

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radiation.* In the present paper no account is taken of the reflecting power of the earth's surface. The tolerable agreement between the results of observation and those of a theory based on a non-reflecting surface shows that for an ordinary landscape the effect of reflection from field and foliage need not be as great as is sometimes supposed.† The effect of snow on the polarization of sky radiation is well-known‡: on this point Lord RAYLEIGH mentions in his 1871 paper the interest which would be attached to sky radiation observations taken over a landscape covered with snow or over the sea, reflection from these surfaces being in both cases especially determinate.§

For meteorological purposes it is important to know the total solar radiation incident on a horizontal plane as well as the contribution due to sky radiation. The results are given in calories per square centimetre per minute for Mount Wilson and Washington in Tables VII. and IX., and are shown graphically in Diagrams IX. and X. It will be noticed that for large zenith distances of the sun the contribution of sky radiation to the total radiation on a horizontal plane is a very considerable fraction of that due to direct solar radiation. The factor of sky radiation is thus of considerable importance in the meteorology of northern latitudes.

The agreement between the results of calculations and such observations as are available gives rise to a hope that the present communication may serve as a guide towards systematic observations of the type dealt with, and to their interpretation in terms of a theory of scattering and absorption; by this means one may hope to obtain absorption constants and methods of using them which will be of some service to meteorology and astrophysics.

Summary.

The analysis of the present paper seems to support the view that at levels above Mount Wilson molecular scattering is sufficient to account completely both for

* In the "Report on the Astrophysical Observatory" (C. G. ABBOT, 'Annual Report of the Smithsonian Institution,' 1911, p. 65), the Director announces that sky radiation observations have been successfully taken at Mount Whitney (August, 1910). Since the transmission coefficients at the time of observation are also determined, the results will enable an accurate comparison of sky radiation results to be made with the values obtained by calculation in terms of the coefficients of attenuation.

† NICOLS, "Theories of the Colour of the Sky," 'Phys. Rev.,' XXVI., June, 1908, p. 507.

‡ Observations have been made by McCONNEL, J. C., on the effect of the nature of the ground on the degree of polarization of the sky at 90 degrees from the sun; the effect of a covering of snow is to diminish the degree of polarization. (PERNTNER, 'Meteorological Optics,' Part IV., 1910, p. 643. The above reference is taken from 'A History of the Cavendish Laboratory,' Longmans, Green & Co., 1910, p. 129.)

§ The analysis of the present paper could be extended without difficulty to take into account the effect of reflection from the earth's surface on the intensity and polarization of sky radiation, provided the landscape were covered by a layer of uniform substance such as snow, or observations were taken over the sea, which presents a determinate reflecting surface.

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attenuation of solar radiation and for the intensity and quality of sky radiation. Even at sea-level the effect of "atmospheric dust" can be taken into account in a simple manner in the formulæ for absorption and scattering. Should future observations support the validity of the simple law expressed by equation (2) connecting the coefficient of attenuation with the wave-length, we may with considerable assurance make use of the law to obtain the coefficients for very short or very long wave-lengths when the direct method of calculation from high and low sun observations leaves room for considerable uncertainty owing to the small intensities in the solar spectrum at these wave-lengths and owing to other experimental difficulties.

SCHUSTER* points out in this connection the extreme importance of determining accurately the form of the solar intensity curve outside the earth's atmosphere for short wave-lengths, since the effect of a solar atmosphere in absorbing and scattering radiation is to give rise to an intensity-curve which does not agree with that given by PLANCK'S formula especially for short wave-lengths.

Absorption and scattering of radiation by the sun's atmosphere, taken in conjunction with effects of self-illumination, constitutes a problem analogous to that just considered for the earth's atmosphere. By making a comparison between the calculated variation of intensity of radiation of different wave-lengths over the solar disc and the results of observation it will be possible to determine from the intensity curve of the normal solar spectrum outside the earth's atmosphere the intensity-curve at the radiating layer of the sun. This corrected curve may then be compared with that given by PLANCK'S formula and a closer approximation made to the temperature of the sun than the values now given. This investigation the writer hopes to be able to deal with in a future communication

* SCHUSTER, *loc. cit.*

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Numerical Tables and Comparison of Theory with Observation.

Discussion of Results.

Table I.—Coefficients of attenuation for Washington, Mount Wilson, Mount Whitney, and Potsdam.

" II.—Constants of atmospheric scattering and absorption.

" III.—Tabulation of the function $Ce^{-C}\,G\{C(\sec\zeta - 1)\}$.

" IV.—Tabulation of the function $G(C\sec\zeta)$.

" V.—Tabulation of the auxiliary functions $\Phi(C, 0)$, $G(C)$, $f(C)$.

" VI.—Relative intensities from zenith sky calculated from Mount Wilson observations.

" VII.—Total solar and sky radiation calculated from Mount Wilson observations.

" VIII.—Relative intensities from zenith sky calculated from Washington observations.

" IX.—Total solar and sky radiation calculated from Washington observations.

" X.—Quality of sky radiation at Mount Wilson, October 17, 1906.

" XI.—Total sky radiation at Mount Wilson, October 19, 1906.

" XII.—Average intensities of sky radiation, Mount Wilson.

" XIII.—Polarization of sky radiation calculated from Washington and Mount Wilson observations.

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TABLE I.—*Coefficients of Attenuation for Washington, Mount Wilson, Mount Whitney, and Potsdam.*

The table of constants of solar intensities and coefficients of attenuation for Washington, Mount Wilson, and Mount Whitney is taken from the revised reductions based on the most recent observations of the Smithsonian Astrophysical Observatory.* The unit expressing the relative intensities of solar radiation between wave-lengths λ and $\lambda + d\lambda$ is arbitrary, and can be reduced to calories per square centimetre per minute by multiplying by a factor η such that

$$\eta \int_0^\infty \mathrm{S}\, d\lambda = 1 \cdot 922 \text{ calories per square centimetre per minute.}\dagger$$

In terms of the unit of intensity given in the table, and taking $d\lambda = \cdot 1\mu$, we find by graphical integration $\int_0^\infty \mathrm{S}\, d\lambda = 3321 \cdot 0$, so that $\eta = \cdot 000578$.

No account is taken of the water-vapour bands which occur mainly in the infra-red, where the intensity of solar radiation is comparatively small. The position and extent of these bands is shown in the curve of the intensities in the normal solar spectrum given in Vol. II. of the 'Annals.'‡

The transmission coefficients for Potsdam are those given by MÜLLER.§

* ABBOT, C. G., "The Sun's Energy-Spectrum and Temperature," 'Astrophysical Journal,' XXXIV., October, 1911, p. 197.

† ABBOT, C. G., and FOWLE, F. E., "The Value of the Solar Constant of Radiation," 'Astrophysical Journal,' XXXIII., April, 1911, p. 191.

‡ 'Annals,' vol. II., p. 104.

§ MÜLLER, G., 'Die Photometrie der Gestirne,' Leipzig, 1897, p. 138.

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TABLE I.

Wave-length in microns, λ.	Mean solar intensity, S.	Per-centage probable error.	Mean coefficients of atmospheric transmission, e^{-C_x}. Washington, 1902–07. e^{-C_x}.	Washington, 1902–07. C_x.	Mount Wilson, 1909–10. e^{-C_x}.	Mount Wilson, 1909–10. C_x.	Mount Whitney, 1909–10. e^{-C_x}.	Mount Whitney, 1909–10. C_x.
0·30 μ	(440)	—	—	—	—	—	(0·510)	(·691)
0·325	1285	30·	—	—	0·(550)	(·598)	0·584	·538
0·35	2700	7·3	—	—	0·612	·491	0·660	·416
0·375	3464	2·7	—	—	0·662	·412	0·738	·304
0·39	3620	1·7	0·445	·810	0·694	·365	0·763	·270
0·42	5261	1·3	0·586	·534	0·764	·269	0·806	·216
0·43	5340	1·7	0·600	·511	0·778	·251	0·822	·196
0·45	6047	1·4	0·640	·446	0·800	·223	0·851	·161
0·47	6253	1·8	0·671	·399	0·827	·190	0·880	·128
0·50	6064	1·9	0·705	·350	0·858	·153	0·900	·105
0·55	5627	2·1	0·739	·302	0·876	·132	0·918	·0866
0·60	5047	2·1	0·760	·274	0·890	·117	0·934	·0683
0·70	3650	0·4	0·839	·176	0·942	·0597	0·956	·0450
0·80	2672	1·2	0·865	·145	0·964	·0367	0·972	·0284
1·00	1664	0·7	0·901	·104	0·973	·0274	0·980	·0202
1·30	897	0·7	0·916	·0876	0·972	·0284	0·980	·0202
1·60	526	1·4	0·930	·0726	0·975	·0253	0·978	·0222
2·00	245	2·4	0·909	·0943	0·957	·0439	0·940	·0619
2·50	43	4·8	0·870	·139	0·(900)	(·105)	0·930	·0726
3·00	(12)	45·	—	—	—	—	0·910	·0943

Mean transmission coefficients for Potsdam. λ.	e^{-C_x}.	C_x.	λ.	e^{-C_x}.	C_x.
·44μ	0·706	·348	·58μ	0·830	·186
·46	0·740	·301	·60	0·840	·174
·48	0·764	·269	·62	0·850	·162
·50	0·781	·247	·64	0·861	·150
·52	0·795	·229	·66	0·871	·138
·54	0·808	·213	·68	0·881	·127
·56	0·819	·200			

Coefficients of Attenuation for Washington, Mount Wilson, Mount Whitney, and Potsdam.

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TABLE II.—*Constants of Atmospheric Scattering and Absorption.*

The coefficients of attenuation given in Table I. for various stations were plotted on a large scale diagram against λ^{-4} as abscissæ. The results are shown in Diagram I. From the large-scale drawing the slopes of the straight lines passing through the mean position of the observed points were obtained and the values of $\tan\theta'$ and $\tan\theta''$ corresponding to short- and long-wave radiation were calculated.* From a knowledge of the mean barometric pressures at these stations the values of β as defined in (75) were calculated for short-wave radiation, leading by (84) to an estimate of N_0, the number of molecules per cubic centimetre of a gas at 0° C. and 760 mm. pressure. The values of the absorption constants Γ' and Γ'' of (80) and (82), which include the effect of "dust" especially noticeable at low-level stations, are also given in the table. At levels higher than Mount Wilson the atmosphere is comparatively dust-free and $\Gamma' = \Gamma'' = \gamma$. This value of γ can then be employed in (78) to give a numerical estimate of molecular absorption.

[* If greater accuracy is required the lines of closest fit to the system of observed points can be drawn by calculating in each case the position of the major axis of inertia of the corresponding system of material points of equal weight. Formulæ for the determination of this line are given by KARL PEARSON ('Phil. Mag.,' vol. II., 6th series, November, 1901, p. 559), and also by SNOW, E. C. ('Phil. Mag.,' March, 1911).—*Note added December* 31, 1912.]

TABLE III.—*Tabulation of the Function*, $Ce^{-C}G\{C(\sec\zeta - 1)\}$.

The function $G(x) = (1-e^{-x})/x$ is tabulated by W. LASH MILLER and T. R. ROSEBRUGH* in a set of extensive tables of the integrals

$$\int_x^\infty u^{-2}e^{-u}\,du, \qquad \int_x^\infty u^{-1}e^{-u}\,du = Ei(-x), \qquad \int_x^\infty ue^{-u}\,du, \qquad \int_x^\infty u^2e^{-u}\,du.$$

Values are given to 9 significant figures at intervals of ·001 between $x = 0$ and $x = 1$, and at intervals of ·01 between $x = 1$ and $x = 2$. From these tables the double-entry Table III. was easily constructed. The calculations of the present table, as well as those of the other tables, were performed on a slide-rule, so that their limit of accuracy is about one or two parts in a thousand.

* W. LASH MILLER and T. R. ROSEBRUGH, 'Trans. Roy. Soc. of Canada,' 2nd series, vol. IX., 1903, sect. iii., pp. 73–107.

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TABLE II.

Station.	Height above sea-level.	Mean barometric pressure, p.	Short-wave radiation, $\lambda < \cdot 610\mu$.				Long-wave radiation, $\lambda > \cdot 610\mu$.		
			tan θ'.	β.	N_0.	Γ'.	tan θ''.	Γ''.	$\beta/\gamma = \beta''/\gamma''$.
	metres.	mm.							
Potsdam	100	*752	·00893	·00904	$2 \cdot 51 \times 10^{19}$	·105	·0170	·054	·32
Washington . .	10	†763·7	·01016	·01011	$2 \cdot 24 \times 10^{19}$	·192	·0277	·088	·32
Mount Wilson .	1780	‡617	·00806	·00994	$2 \cdot 28 \times 10^{19}$	·032	·00806	·032	·32
Mount Whitney	4420	§446·7	·00592	·01003	$2 \cdot 26 \times 10^{19}$	·032	·00592	·032	·32

Constants of Atmospheric Scattering and Absorption.

Authorities:—

* The mean barometric pressure at Potsdam is given by MÜLLER, *loc. cit.*, p. 138.

† The mean annual barometric pressure at Washington is taken from BARTHOLOMEW's 'Atlas of Meteorology,' 1899.

‡ The mean barometric pressure at Mount Wilson is not given explicitly in the 'Annals.' The value given in the above table is obtained by finding the reduction of pressure to sea-level corresponding to an elevation of 5886 feet (the air-temperature being taken at 60° F.) from HAZEN's Tables. ('Professional Papers of the Signal Service,' No. VI., Washington, 1882.)

§ The barometric pressure at Mount Whitney is obtained from observations given by LANGLEY, September 2–6, 1881 (LANGLEY, "Researches on Solar Heat: A Report of the Mount Whitney Expedition," 'Professional Papers of the Signal Service,' No. XV., 1884.)

TABLE III.

C. \ ζ.	0°.	20°.	40°.	60°.	70°.	80°.
0·00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
·05	·0476	·0475	·0472	·0464	·0443	·0424
·10	·0905	·0902	·0891	·0861	·0813	·0720
·14	·1217	·1212	·1191	·1133	·1072	·0889
·18	·1502	·1492	·1461	·1377	·1270	·1009
·22	·1766	·1753	·1708	·1583	·1439	·1093
·26	·2002	·1985	·1927	·1761	·1575	·1150
·30	·2220	·2200	·2136	·1953	·1749	·1174
·34	·2420	·2395	·2298	·2052	·1776	·1198
·38	·2600	·2568	·2450	·2163	·1844	·1206
·50	·303	·298	·280	·238	·1964	·1154
·60	·329	·323	·301	·248	·1950	·1090
·70	·348	·340	·313	·250	·1930	·1006
·80	·359	·350	·318	·247	·1850	·0923
·90	·366	·356	·320	·241	·1740	·0856

Table of the Function $Ce^{-C} G \{C (\sec \zeta - 1)\}$.

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TABLE IV.—*Tabulation of the Function* G (C sec ζ).

This function occurs in calculating the mean coefficient, ϵ, in the solution of the integral equation (44). It is easily tabulated from LASH MILLER'S tables.

TABLE V.—*Tabulation of the Auxiliary Functions* Φ (C, 0), G (C), f (C).

The function $f(x)$ defined by the relation

$$f(x) = x\int_x^{\infty} u^{-2}e^{-u}\,du = e^{-x} + xEi(-x)$$

occurs in the theory of absorption in a flat plate which itself absorbs radiation from a uniform distribution of radiating elements throughout its own volume. A short table of the function $f(x)$ has been given by the writer in a previous paper.*

We write as in (51),

$$B(x) = Ei(-x) - \log x, \qquad B(-x) = Ei(x) - \log x.$$

The function B (x) is given in LASH MILLER'S Tables.† The function B $(-x)$ was calculated by making use of the tables of the exponential integral $Ei(x)$.‡

From these a table of the function

$$\Phi(C, 0) = (1 - e^{-C})\{1 - f(C)\} + B(C) - B(2C) + e^{-C}\{B(C) - \gamma\}$$

was constructed.

* KING, L. V., "Absorption Problems in Radioactivity," 'Phil. Mag.,' February, 1912, p. 245.

† W. LASH MILLER and T. R. ROSEBRUGH, *loc. cit.*, p. 81.

‡ *Cf.* DALE, 'Five-Figure Tables of Mathematical Functions,' Arnolds, London, 1908.

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TABLE IV.

C. \ ζ.	0°.	20°.	40°.	60°.	70°.	80°.
0·00	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
·05	·975	·974	·968	·952	·929	·873
·10	·952	·948	·940	·906	·868	·770
·14	·933	·929	·913	·872	·820	·688
·18	·916	·910	·891	·840	·778	·623
·22	·897	·892	·869	·809	·737	·567
·26	·880	·874	·848	·780	·700	·536
·30	·864	·856	·828	·752	·666	·476
·34	·848	·839	·808	·726	·633	·438
·38	·832	·822	·788	·701	·603	·406
·50	·788	·777	·735	·632	·526	·328
·60	·751	·740	·694	·583	·472	·281
·70	·718	·704	·656	·538	·428	·243
·80	·688	·672	·622	·499	·386	·215
·90	·658	·642	·588	·464	·357	·192

Table of the Function G (C sec ζ).

TABLE V.

C.	Φ (C, 0).	G (C).	f (C).	$\frac{1}{2}$ {G (C) + f (C)}.	$\frac{1}{2}$ {1 + f (C)}.	$f\left(\frac{C}{2}\right)$.
0·00	0·0000	1·0000	1·0000	1·000	1·000	1·000
·05	·0095	·9754	·828	·902	·914	·897
·10	·0311	·9516	·722	·837	·861	·828
·14	·0536	·933	·656	·794	·828	·782
·18	·0801	·916	·600	·758	·800	·741
·22	·1081	·897	·550	·724	·775	·705
·26	·1403	·880	·508	·694	·754	·672
·30	·1723	·864	·469	·667	·735	·641
·34	·2065	·848	·435	·642	·718	·613
·38	·2414	·832	·404	·618	·702	·587
·50	·349	·787	·327	·557	·664	·518
·60	·438	·752	·276	·514	·638	·469
·70	·525	·719	·235	·477	·618	·427
·80	·612	·688	·201	·445	·601	·389
·90	·686	·659	·172	·416	·586	·356

Table of Auxiliary Functions Φ (C, 0), G (C), f (C).

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TABLE VI.—*Relative Intensities from Zenith Sky Calculated from Mount Wilson Observations in Terms of Mean Attenuation Coefficients.*

The wave-lengths at Mount Wilson level corresponding to the tabulated values of the attenuation coefficients C were determined from the formula (76), $C = \beta\lambda^{-4}+\gamma$, making use of the constants β and γ given in Table II., which themselves are derived from observations on the transmission of solar-radiation.

A large scale chart of the intensities in the normal solar spectrum outside the atmosphere was prepared from the data reproduced in Table I. The values of S corresponding to various values of λ were then estimated from this curve of solar intensities. Auxiliary tables of the coefficients ϵ_2, ϵ, ϵ_1 were calculated from (44), making use of Table V. In this way a table of the function

$$\tfrac{1}{2}\frac{c}{C}\begin{pmatrix}\epsilon_2\\ \epsilon\\ \epsilon_1\end{pmatrix}\Phi(C, 0)$$

was prepared. This table, with Table III., enabled the term in square brackets in the expression for $T(0, \zeta)$ to be calculated. Finally, making use of the values of S just determined, the present table of the intensity from zenith sky, $T(0, \zeta)$, was calculated. The extreme and mean solutions of the integral equation are retained throughout. It will be noticed that the extreme solutions diverge rapidly for large values of C and ζ; it must be remembered that for large values of C the value of S decreases with extreme rapidity, no appreciable intensity having been measured for a wave-length less than $\cdot 3\mu$. Consequently, in calculating the total intensity of sky radiation on a horizontal surface, the divergence of the extreme solutions does not lead to a very great divergence in the final results. The results given in the present table are shown graphically in Diagrams II.–VII.

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TABLE VI.

C.	λ.	S.	0°.	20°.	40°.	60°.	70°.	80°.
·05	·741μ	3140	*9·8* 9·8 **9·8**	*9·2* 9·2 **9·2**	*7·7* 7·7 **7·7**	*6·0* 6·0 **6·0**	*5·1* 5·1 **5·1**	*4·6* 4·6 **4·6**
·10	·571	5380	*50·9* 50·6 **50·2**	*47·7* 47·4 **47·0**	*39·8* 39·5 **39·0**	*30·4* 30·1 **29·6**	*25·9* 25·4 **24·8**	*21·6* 20·8 **20·0**
·14	·510	6000	*88·5* 87·2 **85·9**	*83·0* 81·7 **80·3**	*69·0* 67·7 **66·4**	*52·2* 50·7 **49·4**	*44·6* 42·8 **41·3**	*35·6* 32·9 **31·2**
·18	·474	6260	*124·3* 121·7 **118·3**	*116·7* 113·8 **110·2**	*96·5* 93·7 **90·7**	*72·8* 69·8 **66·1**	*61·2* 57·5 **54·0**	*48·0* 42·3 **38·1**
·22	·446	5910	*146·0* 141·5 **136·8**	*137·0* 132·5 **127·8**	*113·0* 108·4 **104·0**	*84·5* 79·8 **75·1**	*70·1* 64·5 **59·6**	*53·9* 45·4 **40·0**
·26	·427	5320	*166·9* 159·0 **151·0**	*156·0* 148·5 **140·8**	*128·5* 121·2 **114·1**	*95·5* 87·7 **80·6**	*79·4* 70·1 **62·5**	*60·7* 48·0 **39·4**
·30	·410	4480	*168·0* 158·4 **148·5**	*157·1* 147·9 **137·9**	*130·0* 120·6 **108·2**	*97·0* 86·6 **78·0**	*80·5* 69·0 **60·3**	*60·0* 43·7 **34·6**
·34	·397	4000	*173·2* 161·0 **148·1**	*162·0* 150·0 **137·1**	*133·1* 121·3 **110·0**	*98·4* 85·6 **75·1**	*80·8* 66·5 **55·8**	*61·5* 41·6 **31·5**
·38	·385	3720	*184·1* 168·0 **152·0**	*171·9* 156·0 **140·9**	*141·0* 126·1 **111·6**	*103·8* 87·9 **75·2**	*85·1* 67·1 **54·6**	*65·1* 41·0 **29·5**
·50	·357	2960	*205·0* 178·0 **151·7**	*191·5* 165·0 **140·0**	*156·4* 131·0 **108·8**	*115·3* 89·6 **69·5**	*94·5* 65·2 **47·6**	*73·1* 36·4 **22·5**
·60	·341	2140	*185·5* 153·0 **123·8**	*173·1* 141·4 **113·7**	*141·8* 111·3 **87·0**	*103·4* 73·0 **52·8**	*84·0* 52·1 **34·3**	*68·0* 26·5 **15·1**

Relative Intensities from Zenith Sky Calculated from Mount Wilson Observations in Terms of Mean Attenuation Coefficients.

$$T(0, \zeta) = \frac{1}{4\pi}\{\tfrac{3}{4}(1+\cos^2\zeta)\}\frac{c}{C}S\left[Ce^{-C}G\{C(\sec\zeta-1)\}+\tfrac{1}{2}\frac{c}{C}\begin{pmatrix}\epsilon_2\\ \epsilon\\ \epsilon_1\end{pmatrix}\Phi(C, 0)\right]$$

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TABLE VII.—*Total Solar and Sky Radiation Calculated at Mount Wilson Level from Mean Attenuation Coefficients.*

In order to obtain the total solar intensity reaching the earth's surface for different zenith distances of the sun, a table of the values of $Se^{-C\sec\zeta}$ was drawn up for the various values of C, λ, S, and ζ given in Table VI. From these curves were drawn on a large scale, which, when integrated, gave the values of

$$\int_0^\infty E(\zeta)\,d\lambda = \int_0^\infty Se^{-C\sec\zeta}\,d\lambda,$$

the units of intensity being those given for S in Table I., and the unit of wave-length being taken as 1μ, and $d\lambda = \cdot 1\mu$. As a check on the calculations it is interesting to see how far the total intensity can be represented by a formula of the type

$$\int_0^\infty E(\zeta)\,d\lambda = e^{-\bar{C}\sec\zeta}\int_0^\infty S\,d\lambda.$$

The values of the "apparent" coefficient of attenuation $\bar{C}$ calculated from the integrated area of each of the curves of solar intensity corresponding to different values of ζ are given below :—

ζ.	0°.	20°.	40°.	60°.	70°.	80°.	Mean values $\bar{C}$ between 0° and 60°.
$\bar{C}$	·1023	·1140	·1293	·1079	·0953	·0817	$\bar{C} = \cdot 1134$, $e^{-\bar{C}} = \cdot 893$.

The value of the apparent transmission of total radiation given for Mount Wilson* is $e^{-\bar{C}} = \cdot 895$, $\bar{C} = \cdot 111$, in good agreement with the above values. It will be noticed that an exponential formula for the total intensity fails for greater zenith distances than 60°.

From the intensities from zenith sky given in Table VI., and drawn in Diagrams II.–VII., the integrals $\int_0^\infty T(0,\zeta)\,d\lambda$ were calculated for various values of ζ. From these the total sky radiation on a horizontal plane, $H(\zeta)$, was calculated from the approximate formula (68), and is plotted against values of ζ in Diagram IX. It will be noticed that the value of the ratio

$$\int_0^\infty H(\zeta)\,d\lambda \Big/ \int_0^\infty E(\zeta)\,d\lambda$$

is very nearly constant for all zenith distances and has the mean value ·050. This value is in fair agreement with the value ·052 (August 18, 1905) but is considerably smaller than the value ·077 (September 8, 1906, October 19, 1906) determined at Mount Wilson.† It will be noticed under Table XII. that the attenuation coefficients for August 18, 1905, are in much better agreement with the mean values employed in these calculations than those for October 19, 1906.

The entries in the last row of the present table give the ratio *sky/sun* for equal solid angles, the semi-diameter of the sun being taken as 16′ of arc. These values are those most often measured in observations on sky radiation ; the results are shown graphically in Diagram XI.

* 'Annals,' vol. II., p. 96.

† 'Annals,' vol. II., Table 35, p. 153.

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TABLE VII.

	Units.	0°.	20°.	40°.	60°.	70°.	80°.
Total Solar Radiation per unit area normal to sun's rays reaching earth's surface. Values of $\int_0^\infty E(\zeta)\, d\lambda$ $\int_0^\infty S\, d\lambda = 3321$ (S, λ) units $= 1\cdot922$ calories per sq. cm. per minute.	(S, λ) units $d\lambda = \cdot 1\mu$.	3000·5	2944·5	2782·5	2673·0	2512·0	2018·5
	Calories per square centimetre per minute.	1·740	1·706	1·623	1·550	1·457	1·169
Total Solar Radiation per unit area of horizontal plane reaching earth's surface. Values of $\cos\zeta \int_0^\infty E(\zeta)\, d\lambda$.	(S, λ) units $d\lambda = \cdot 1\mu$.	3000·5	2770	2130	1337	858	350
	Calories per square centimetre per minute.	1·740	1·602	1·242	·775	·498	·203
Total Radiation per unit area of horizontal surface from unit solid angle of zenith sky. Values of $\int_0^\infty T(0, \zeta)\, d\lambda$.	(S, λ) units $d\lambda = \cdot 1\mu$.	*33·85* 36·15 **38·05**	*32·2* 34·4 **36·6**	*25·5* 27·8 **30·0**	*18·3* 20·3 **22·6**	*13·4* 15·9 **18·6**	*9·8* 11·4 **14·9**
	Calories per square centimetre per minute.	*·0196* ·0209 **·0220**	*·0186* ·0199 **·0212**	*·0147* ·0161 **·0173**	*·0106* ·0117 **·0131**	*·0077* ·0092 **·0107**	*·0057* ·0066 **·0086**
Total Sky Radiation per unit area of horizontal surface. Values of $\int_0^\infty H(\zeta)\, d\lambda = \frac{2\pi}{\frac{3}{4}(1+\cos^2\zeta)} \int_0^\infty T(0,\zeta)\, d\lambda$.	(S, λ) units $d\lambda = \cdot 1\mu$.	*141·9* 151·2 **159·3**	*143·2* 153·1 **163·0**	*134·8* 147·0 **158·6**	*122·2* 135·9 **151·1**	*100·6* 119·2 **139·6**	*79·4* 92·5 **120·9**
	Calories per square centimetre per minute.	*·0824* ·0877 **·0925**	*·0832* ·0888 **·0945**	*·0782* ·0853 **·0920**	*·0710* ·0788 **·0876**	*·0584* ·0692 **·0809**	*·0461* ·0537 **·0701**
Total Radiation on horizontal surface $\int_0^\infty \{\cos\zeta\, E(\zeta) + H(\zeta)\}\, d\lambda$.	Calories per square centimetre per minute.	*1·822* 1·828 **1·833**	*1·685* 1·691 **1·697**	*1·320* 1·327 **1·334**	*·846* ·854 **·863**	*·556* ·567 **·597**	*·249* ·257 **·273**
Ratio $\frac{\int_0^\infty H(\zeta)\, d\lambda}{\cos\zeta \int_0^\infty E(\zeta)\, d\lambda}$.	—	*·0473* ·0504 **·0531**	*·0517* ·0553 **·0588**	*·0633* ·0690 **·0744**	*·0915* ·1018 **·1131**	*·1171* ·1390 **·1623**	*·227* ·264 **·346**
Ratio $\frac{\int_0^\infty H(\zeta)\, d\lambda}{\int_0^\infty E(\zeta)\, d\lambda}$.	—	*·0473* ·0504 **·0531**	*·0487* ·0520 **·0553**	*·0485* ·0529 **·0570**	*·0453* ·0508 **·0565**	*·0400* ·0475 **·0555**	*·0394* ·0458 **·0597**
Ratio $\omega \frac{\int_0^\infty T(0,\zeta)\, d\lambda}{\int_0^\infty E(\zeta)\, d\lambda}$. $\omega = 2\pi(1 - \cos 16') = 6\cdot80 \times 10^{-5}$	—	*76·6 × 10⁻⁸* 82·0 **86·2**	*74·3 × 10⁻⁸* 79·5 **84·5**	*62·2 × 10⁻⁸* 67·9 **73·2**	*46·6 × 10⁻⁸* 51·7 **57·5**	*36·3 × 10⁻⁸* 43·1 **50·3**	*33·0 × 10⁻⁸* 38·4 **50·2**

Total Solar and Sky Radiation Calculated at Mount Wilson Level from Mean Attenuation Coefficients.

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE VIII.—*Relative Intensities from Zenith Sky Calculated from Washington Observations in Terms of Mean Attenuation Coefficients.*

This table is constructed on the same plan as Table VI. for Mount Wilson, making use of the attenuation constants given in Table II. for Washington. The effect of "dust" requires us to make use of different constants for long- and for short-wave radiation in calculating wave-lengths corresponding to the tabulated values of C from the formula $C = \beta\lambda^{-4}+\gamma$. The result is a discontinuity in the sky radiation curve in the neighbourhood of $\cdot 610\mu$, shown in the curves of Diagram XII. The mean value of the solution of the integral equation is alone given, since the values of the absorption coefficients vary so rapidly from day to day that the intensities of sky radiation calculated from them probably differ by more than the difference between the extreme solutions.

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TABLE VIII.

C.	λ.	S.	0°.	20°.	40°.	60°.	70°.	80°.
·10	1·08 μ	1360	3·04	2·85	2·37	1·81	1·54	1·25
·14	·824	2470	16·2	15·2	12·5	9·44	7·85	6·11
·18	·725	3350	38·5	36·1	29·9	22·15	18·23	13·4
·22	·654	4240	69·0	64·9	52·8	38·70	31·40	22·1
·26	·625	4640	97·5	91·1	74·4	53·8	42·9	29·3
·30	·556	5460	58·5	54·5	44·5	32·1	25·6	16·0
·34	·509	6020	88·0	83·0	67·1	47·3	36·4	22·8
·38	·482	6230	127·1	109·0	88·0	61·1	46·5	28·4
·50	·426	5320	163·1	150·9	119·4	80·5	59·4	32·8
·60	·398	4320	173·0	160·0	126·0	82·4	57·3	30·8
·70	·376	3460	169·8	156·0	121·8	77·5	53·9	27·0
·80	·360	2900	166·9	153·1	118·2	73·6	49·7	22·8
·90	·346	2240	146·0	133·8	102·6	62·0	41·3	19·5

Relative Intensities from Zenith Sky Calculated from Washington Observations in Terms of Mean Attenuation Coefficients.

$$T(0,\zeta) = \frac{1}{4\pi}\{\tfrac{3}{4}(1+\cos^2\zeta)\}\frac{c}{C}S\left[Ce^{-C}G\{C(\sec\zeta-1)\}+\tfrac{1}{2}\frac{c}{C}\epsilon\Phi(C,0)\right].$$

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TABLE IX.—*Total Solar and Sky Radiation Calculated at Washington Level from Mean Attenuation Coefficients.*

This table is drawn up on the same plan as Table VII., making use of the Washington data given in Table II. The values of the "apparent" coefficients of attenuation, $\bar{C}$, derived from the integrated solar radiation curves corresponding to different zenith distances of the sun are given below:—

ζ.	0°.	20°.	40°.	60°.	70°.	80°.	Mean value $\bar{C}$ between 0° and 60°.
$\bar{C}$	·247	·247	·252	·246	·235	·210	$\bar{C}$ = ·248, $e^{-\bar{C}}$ = ·780

The value of the apparent transmission of total radiation given for Washington* is $e^{-\bar{C}} = \cdot 787$, $\bar{C} = \cdot 240$ in substantial agreement with the above mean value between 0° and 60°.

The ratio

$$\int_0^\infty H(\zeta)\,d\lambda \Big/ \int_0^\infty E(\zeta)\,d\lambda$$

is approximately constant for all zenith distances to 70° and has the mean value ·078. The value of this ratio for $\zeta = 80°$ seems to indicate the existence of a rapidly increasing value of the ratio beyond that angle: the analysis by which sky radiation is calculated ceases to hold even approximately for greater zenith distances on account of the curvature of the earth.

* 'Annals,' vol. II., p. 96.

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TABLE IX.

	Units.	0°.	20°.	40°.	60°.	70°.	80°.
Total Solar Radiation per unit area normal to sun's rays reaching earth's surface. Values of $\int_0^\infty E(\zeta)\,d\lambda$	(S, λ) units $d\lambda = \cdot 1\mu$.	2597·5	2553·0	2389·0	2032·5	1670·0	991·0
$\int_0^\infty S\,d\lambda = 3321$ (S, λ) units $= 1\cdot922$ calories per sq. cm. per minute.	Calories per square centimetre per minute.	1·502	1·478	1·382	1·177	·967	·574
Total Solar Radiation per unit area of horizontal plane reaching earth's surface. Values of	(S, λ) units $d\lambda = \cdot 1\mu$.	2597·5	2400	1830	1016	571	172
$\cos\zeta \int_0^\infty E(\zeta)\,d\lambda$.	Calories per square centimetre per minute.	1·502	1·390	1·060	·589	·331	·100
Total Radiation per unit area of horizontal surface from unit solid angle of zenith sky. Values of	(S, λ) units $d\lambda = \cdot 1\mu$.	47·5	44·1	34·9	24·1	17·4	10·6
$\int_0^\infty T(0,\zeta)\,d\lambda$.	Calories per square centimetre per minute.	·0275	·0255	·0202	·0140	·0101	·0062
Total Sky Radiation per unit area of horizontal surface. Values of	(S, λ) units $d\lambda = \cdot 1\mu$.	199	197	184	161	130	86
$\int_0^\infty H(\zeta)\,d\lambda = \frac{2\pi}{\frac{3}{4}(1+\cos^2\zeta)} \int_0^\infty T(0,\zeta)\,d\lambda$.	Calories per square centimetre per minute.	·115	·114	·1065	·0931	·0753	·0497
Total Radiation on horizontal surface $\int_0^\infty \{\cos\zeta\,E(\zeta) + H(\zeta)\}\,d\lambda$.	Calories per square centimetre per minute	1·617	1·504	1·166	·682	·406	·150
Ratio $\frac{\int_0^\infty H(\zeta)\,d\lambda}{\cos\zeta \int_0^\infty E(\zeta)\,d\lambda}$.	—	·0778	·0821	·1005	·1588	·228	·506
Ratio $\frac{\int_0^\infty H(\zeta)\,d\lambda}{\int_0^\infty E(\zeta)\,d\lambda}$.	—	·0778	·0773	·0766	·0793	·0779	·0865
Ratio $\omega \frac{\int_0^\infty T(0,\zeta)\,d\lambda}{\int_0^\infty E(\zeta)\,d\lambda}$. $\omega = 2\pi(1-\cos 16') = 6\cdot80 \times 10^{-5}$.	—	$124\cdot3 \times 10^{-8}$	$117\cdot6 \times 10^{-8}$	$99\cdot4 \times 10^{-8}$	$80\cdot7 \times 10^{-8}$	$70\cdot8 \times 10^{-8}$	$72\cdot7 \times 10^{-8}$

Total Solar and Sky Radiation Calculated at Sea-Level from Mean Attenuation Coefficients at Washington.

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TABLE X.—*Quality of Sky Radiation at Mount Wilson, October 17, 1906.*

Observations on the quality of sky-radiation have been made at Mount Wilson.* The values given in the present table are calculated in terms of the *mean* coefficients of attenuation at that station. The units of intensity in the calculated and observed values are both arbitrary. The calculated intensities are reduced to the same units as the observed intensities by multiplying the former by the mean value of the ratios of the intensities corresponding to the different wave-lengths. The results are compared graphically in Diagram XIII. The agreement is seen to be fairly satisfactory, in spite of the fact that the *mean* values of the coefficients of attenuation are used in calculating intensities on a particular day.

* 'Annals,' vol. II., p. 155.

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TABLE X.

		·422μ.	·457μ.	·491μ.	·556μ.	·614μ.	·660μ.
(i.) *Quality of Sky Radiation.* Comparison of Calculated Values with Mount Wilson Observations, October 17, 1906. $\zeta = 43°$. $\phi = 23°$. $\psi = 79°$.	Solar intensity, $\zeta = 43°$.	3430	4700	4960	4630	4080	3600
	Zenith sky, $\zeta = 43°$.	*124* 116 **109**	*102* 97 **92**	*83* 80 **78**	*44* 43 **42**	*26* 26 **26**	*18* 18 **18**
	Sky/sun, $\zeta = 43°$.	*361* $\times 10^{-4}$ 339 **318**	*218* $\times 10^{-4}$ 206 **196**	*167* $\times 10^{-4}$ 161 **157**	*95·0* $\times 10^{-4}$ 92·5 **90·5**	*63·5* $\times 10^{-4}$ 63·5 **63·5**	*50* $\times 10^{-4}$ 50 **50**
	Observed *sky/sun.*	655	521	294	188	106	100
	Calculated *sky/sun.*	*722* 678 **636**	*436* 412 **392**	*334* 321 **314**	*190* 185 **181**	*127* 127 **127**	*100* 100 **100**
(ii.) *Quality of Sky Radiation.* Comparison of Calculated Values with Mount Wilson Observations. $\zeta = 58°$. $\phi = 17°$. $\psi = 146°$.	Solar intensity, $\zeta = 58°$.	3050	4210	4570	4360	3850	3420
	Zenith sky, $\zeta = 58°$.	*99·0* 91·0 **82·5**	*81·0* 77·5 **73·5**	*65* 62 **60**	*37* 36 **35**	*20·5* 20·5 **20·5**	*13·5* 13·5 **13·5**
	Sky/sun, $\zeta = 58°$.	*325* $\times 10^{-4}$ 295 **271**	*193* $\times 10^{-4}$ 184 **175**	*142* $\times 10^{-4}$ 136 **131**	*85·0* $\times 10^{-4}$ 82·5 **80·0**	*53·3* $\times 10^{-4}$ 53·3 **53·3**	*39·5* $\times 10^{-4}$ 39·5 **39·5**
	Observed *sky/sun.*	574	425	317	191	124	104
	Calculated *sky/sun.*	*750* 680 **625**	*445* 424 **403**	*327* 314 **302**	*196* 190 **185**	*123* 123 **123**	*91* 91 **91**
(iii.) *Quality of Sky Radiation.* Comparison of Calculated Values of Zenith Sky with Mean Absolute Intensity of Sky Radiation. Mean, $\zeta = 50°$.	Zenith sky.	*112·6* 104·5 **97·0**	*92·0* 88·0 **83·6**	*73·5* 71·5 **69·0**	*40·5* 40·0 **39·5**	*22·2* 22·2 **22·2**	*15·7* 15·7 **15·7**
	Observed mean sky.	1194	986	701	395	231	174
	Calculated zenith sky.	*1192* 1109 **1030**	*975* 933 **885**	*780* 758 **731**	*429* 424 **418**	*235* 235 **235**	*166* 166 **166**

Quality of Sky Radiation at Mount Wilson, October 17, 1906.

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TABLE XI.—*Total Sky Radiation at Mount Wilson, October 19, 1906.*

Observations on the total intensity of sky-radiation are given for Mount Wilson* under the date October 19, 1906. Attenuation coefficients for this day are not given, so that the observations are compared with values calculated from the mean coefficients at Mount Wilson. If ω is the solid angle subtended by the sun, the value of

$$\omega \int_0^\infty \mathrm{T}(\phi, \zeta)\, d\lambda$$

is calculated from the approximate formula (66),

$$\omega \int_0^\infty \mathrm{T}(\phi, \zeta)\, d\lambda = \omega \frac{1+\cos^2\theta}{1+\cos^2\zeta} \sec\phi \int_0^\infty \mathrm{T}(0, \zeta)\, d\lambda.$$

Making use of the values of

$$\omega \int_0^\infty \mathrm{T}(0, \zeta)\, d\lambda \Big/ \int_0^\infty \mathrm{E}(\zeta)\, d\lambda$$

from Table VII., and interpolating for intermediate values of ζ from the curves given in Diagram XI., the ratio

$$\omega \int_0^\infty \mathrm{T}(\phi, \zeta)\, d\lambda \Big/ \int_0^\infty \mathrm{E}(\zeta)\, d\lambda$$

was calculated.

The angle θ between sun and sky was calculated from the formula

$$\cos\theta = \cos\phi\cos\zeta + \sin\phi\sin\zeta\cos\psi,$$

the azimuth ψ being measured from a vertical plane through the sun in the direction N.–E.–S.–W.

The comparison of the calculated ratios with those observed is fairly satisfactory; the greatest discrepancies occur when θ is small and when ζ is large. The probable reasons for this lack of agreement are given at the end of Section 8.

* 'Annals,' vol. II., Table 32, p. 151.

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TABLE XI.

ζ.	ϕ.	ψ.	θ.	$\frac{1+\cos^2\theta}{1+\cos^2\zeta}$.	sec ϕ.	$\frac{\text{Zenith sky}}{\text{Sun}} \times 10^{-8}$.	$\frac{\text{Sky}}{\text{Sun}} \times 10^{-8}$.	Observed Sky/Sun $\times 10^{-8}$.
55°·8	79°·2	− 43°·5	46°·1	1·127	5·34	*60·2* 55·0 **49·0**	*361* 330 **294**	295
55·5	67·4	− 41·2	37·9	1·228	2·60	*60·6* 55·1 **50·0**	*193* 176 **160**	219
54·0	54·5	− 35·5	28·7	1·316	1·72	*61·8* 56·3 **50·5**	*140* 127 **114**	165
53·6	44·6	− 21·3	18·6	1·402	1·40	*62·0* 57·0 **51·0**	*122* 112 **100**	190
50·4	40·5	− 6·7	11·5	1·392	1·31	*64·6* 59·3 **53·2**	*118* 108 **97**	285
49·8	31·1	12·9	20·6	1·327	1·17	*65·0* 60·0 **54·0**	*101* 93 **84**	350
49·2	49·7	− 149·6	94·4	·696	1·55	*65·5* 60·4 **54·5**	*70·7* 65·2 **58·8**	98
48·9	50·4	− 113·3	79·1	·707	1·57	*65·8* 60·6 **54·8**	*73·0* 67·2 **60·8**	108
47·4	79·2	− 63·4	63·2	·826	5·34	*67·0* 61·8 **56·0**	*296* 273 **247**	222
47·0	77·9	− 84·5	77·7	·714	4·77	*67·2* 62·0 **56·2**	*229* 211 **191**	171
46·5	29·2	− 79·6	48·5	·977	1·15	*67·8* 62·5 **57·0**	*76·1* 70·2 **64·0**	98
46·2	27·8	144·8	70·3	·752	1·13	*68·0* 63·0 **57·1**	*57·7* 53·5 **48·5**	68

Total Sky Radiation at Mount Wilson, October 19, 1906.

Comparison of ratio *sky/sun* for equal solid angles with results of calculation from the formula

$$\omega \int_0^\infty \mathrm{T}(\phi, \zeta)\, d\lambda \Big/ \int_0^\infty \mathrm{E}(\zeta)\, d\lambda.$$

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TABLE XII.—*Average Intensities of Sky Radiation, Mount Wilson.*

Observations are given at Mount Wilson* for the average total intensity of sky radiation taken for different azimuths. The theoretical formula corresponding to average sky intensity is by (66),

$$\frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^{\infty} \mathrm{T}(\phi, \zeta)\, d\lambda = \left[\frac{1}{2\pi}\int_0^{2\pi} \frac{1+\cos^2\theta}{1+\cos^2\zeta}\, d\psi\right] \sec\phi \int_0^{\infty} \mathrm{T}(0, \zeta)\, d\lambda.$$

The term in square brackets denoted by

$$\left[\frac{1+\cos^2\theta}{1+\cos^2\zeta}\right]_{\text{av.}} = 1 - \tfrac{1}{2}\sin^2\phi \, . \, \frac{3\cos^2\zeta - 1}{1+\cos^2\zeta}\, .$$

In calculating *average sky/sun* for equal solid angles, the value of

$$\omega \int_0^{\infty} \mathrm{T}(0, \zeta)\, d\lambda \Big/ \int_0^{\infty} \mathrm{E}(\zeta)\, d\lambda$$

is derived by interpolation from Diagram XI.

In order to compare the attenuation coefficients on the dates of the above sky observations, these values taken from the Mount Wilson observations† are compared in a table with the mean values employed in the calculation. It will be noticed that the absorption constants on August 18, 1905, are in much better agreement with the mean values than those of September 8, 1906. The result is that as we should expect the sky observations on the former date agree more closely with the calculated values than those on the latter.

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* 'Annals,' vol. II., Table 32, p. 151.
† 'Annals,' vol. II., pp. 96–97.

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TABLE XII.

ζ.	ϕ.	$\left[\frac{1+\cos^2\theta}{1+\cos^2\zeta}\right]_{\text{av.}}$	sec ϕ.	$\frac{\text{Zenith sky}}{\text{Sun}} \times 10^{-8}$.	$\frac{\text{Av. sky}}{\text{Sun}} \times 10^{-8}$.	Observed Av. sky/Sun $\times 10^{-8}$.
47°·3	29°·4	·969	1·15	*67·0* 62·0 **56·1**	*74·8* 69·2 **62·6**	122
51·2	47·9	·945	1·49	*64·0* 59·0 **53·1**	*90·0* 83·0 **74·7**	129
54·7	64·5	·999	2·24	*61·1* 56·0 **50·0**	*137* 125 **112**	185
50·0	78·8	·918	5·15	*65·0* 60·0 **53·8**	*308* 284 **255**	214
54·5	85·5	·995	12·75	*60·6* 56·0 **50·0**	*770* 710 **635**	500 ?

Sky Radiation, Mount Wilson, September 8, and October 19, 1906.

ζ.	ϕ.	$\left[\frac{1+\cos^2\theta}{1+\cos^2\zeta}\right]_{\text{av.}}$	sec ϕ.	$\frac{\text{Zenith sky}}{\text{Sun}} \times 10^{-8}$.	$\frac{\text{Av. sky}}{\text{Sun}} \times 10^{-8}$.	Observed Av. sky/Sun $\times 10^{-8}$.
27°	20°	·953	1·064	*82·0* 77·0 **71·8**	*83·2* 78·0 **73·0**	78
22	30	·894	1·155	*84·0* 79·0 **74·0**	*87·0* 81·8 **76·6**	82
23	45	·791	1·414	*83·9* 78·8 **73·4**	*94·0* 88·4 **82·2**	67
32	55	·781	1·743	*79·0* 74·4 **69·0**	*108* 102 **94**	81
23	65	·658	2·366	*83·9* 78·8 **73·4**	*131* 123 **114**	93
29	80	·642	5·76	*81·5* 76·0 **70·9**	*301* 281 **262**	116

Sky Radiation, Mount Wilson, August 18, 1905.

Average Intensities of Sky Radiation.

λ.	C, Aug. 18, 1905.	C, Sept. 8, 1906.	C, average.	Date.	$\frac{p}{p_0}\beta$.	$\frac{p}{p_0}\gamma$.
·40μ	·342	·263	·356	Aug. 18, 1905	·0087	·050
·45	·265	·180	·230			
·50	·202	·129	·148	Sept. 8, 1906	·0059	·033
·60	·139	·065	·085			
·70	·081	·036	·070	Mean at Mount Wilson	·0081	·026
·80	·051	·024	·043			
·90	·046	·020	·036			
1·00	·048	·016	·032			
1·20	·045	·014	·028			
1·60	·046	·010	·025			

Attenuation Coefficients on Dates of Sky Observations.

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TABLE XIII.—*Polarization of Sky Radiation Calculated from Mount Wilson and Washington Observations.*

Numerical results on the polarization of sky radiation are calculated from the approximate formulæ given in Section 6. In order to save space numerical values corresponding to the formulæ given at the foot of the table are entered in the same order under each value of λ and ζ.

The values of

$$P(0, \zeta) = Ce^{-C}G\{C(\sec\zeta - 1)\}$$

and

$$Q(0, \zeta) = \tfrac{1}{2}c/C \,.\, \epsilon\,\Phi(C, 0)$$

are obtained from Tables VI. and VIII.

The ratios of the intensities of zenith sky polarized in the principal plane and in a plane at right angles to it are calculated in the form

$$\frac{T_1(0, \zeta)}{T_2(0, \zeta)} = \frac{1 + Q(0, \zeta)/P(0, \zeta)}{\cos^2\zeta + Q(0, \zeta)/P(0, \zeta)}.$$

The results are shown graphically in Diagram XIV.

The ratios $Q(0, \zeta)/P(0, \zeta)$ given in the present table compare the contribution of self-illumination to the sky intensity of different wave-lengths with the effect of direct sunlight on sky radiation.

The ratios

$$\frac{Q(0, \zeta)/P(0, \zeta)}{1 + Q(0, \zeta)/P(0, \zeta)}$$

compare the *residual intensity* of sky radiation over the great circle of most complete polarization 90° from the sun to the intensity polarized in a plane at right angles. It will be noticed that this ratio remains nearly the same for all zenith distances, and is very small for long wave-lengths when the effect of self-illumination is small.

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TABLE XIII.

C.	Mount Wilson.				Washington.			
	λ.	20°.	60°.	80°.	λ.	20°.	60°.	80°.
·05	·741μ	*1·19* ·055 **·052**	*3·47* ·054 **·051**	*18·42* ·054 **·051**	∞	—	—	—
·10	·571	*1·17* ·142 **·124**	*2·91* ·142 **·124**	*6·55* ·145 **·127**	1·08μ	*1·14* ·033 **·032**	*3·64* ·035 **·034**	*15·98* ·035 **·034**
·14	·510	*1·16* ·204 **·170**	*2·63* ·208 **·172**	*5·10* ·207 **·172**	·824	*1·12* ·097 **·088**	*3·16* ·097 **·089**	*8·60* ·098 **·089**
·18	·474	*1·15* ·268 **·212**	*2·44* ·268 **·212**	*4·23* ·271 **·213**	·725	*1·11* ·157 **·136**	*2·84* ·157 **·136**	*6·14* ·159 **·137**
·22	·446	*1·15* ·291 **·225**	*2·39* ·292 **·220**	*3·98* ·296 **·228**	·654	*1·11* ·212 **·175**	*2·62* ·214 **·176**	*4·95* ·217 **·178**
·26	·427	*1·13* ·388 **·280**	*2·18* ·384 **·277**	*3·20* ·411 **·291**	·625	*1·10* ·271 **·213**	*2·44* ·273 **·214**	*4·06* ·287 **·224**
·30	·416	*1·13* ·446 **·308**	*2·09* ·440 **·306**	*2·96* ·464 **·317**	·556	*1·11* ·135 **·119**	*2·95* ·134 **·118**	*6·70* ·140 **·123**
·34	·397	*1·13* ·504 **·335**	*1·99* ·510 **·338**	*2·71* ·526 **·345**	·509	*1·11* ·184 **·155**	*2·75* ·178 **·151**	*5·35* ·193 **·162**
·38	·385	*1·12* ·566 **·362**	*1·91* ·570 **·363**	*2·56* ·594 **·373**	·482	*1·11* ·234 **·190**	*2·54* ·237 **·192**	*4·48* ·249 **·199**
·50	·357	*1·10* ·749 **·429**	*1·74* ·763 **·438**	*2·15* ·814 **·449**	·426	*1·09* ·380 **·275**	*2·18* ·387 **·279**	*3·20* ·413 **·292**
·60	·341	*1·10* ·902 **·474**	*1·63* ·920 **·480**	*1·93* 1·012 **·507**	·398	*1·09* ·504 **·335**	*1·98* ·518 **·341**	*2·62* ·570 **·363**
·70	*Polarization of zenith sky* $\left.\right\} \frac{T_1(0,\zeta)}{T_2(0,\zeta)} = \frac{1+Q(0,\zeta)/P(0,\zeta)}{\cos^2\zeta + Q(0,\zeta)/P(0,\zeta)}$.				·376	*1·08* ·633 **·387**	*1·82* ·660 **·398**	*2·26* ·740 **·425**
·80	Ratio $Q(0,\zeta)/P(0,\zeta)$.				·360	*1·07* ·768 **·435**	*1·70* ·814 **·450**	*2·00* ·945 **·486**
·90	**Polarization of sky 90° from sun** $\left.\right\}$ **Ratio** $\frac{Q(0,\zeta)/P(0,\zeta)}{1+Q(0,\zeta)/P(0,\zeta)}$.				·346	*1·07* ·901 **·474**	*1·62* ·961 **·490**	*1·90* 1·11 **·527**

Polarization of Sky Radiation Calculated from Mount Wilson and Washington Observations.

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DIAGRAM I.—*Variation of Coefficients of Atmospheric Attenuation with Wave-Length.*

The curves of Diagram I. are drawn from the data given in Table I. The coefficients of attenuation C_x at the various stations are plotted against the inverse fourth power of the wave-lengths, the wave-lengths λ being expressed in microns. The observations give rise to a law of the form

$$C_x = \beta\lambda^{-4}+\gamma.$$

The stations to which the various lines refer are as follows:—

Curve I.	Mount Whitney.	*Curve III.*	Potsdam.
Curve II.	Mount Wilson.	*Curve IV.*	Washington.

The conclusions derived from these observations are described in Section 7, and the constants of atmospheric scattering and absorption derived from similar curves plotted on a large scale are given in Table II.

The discontinuity in the curves in the neighbourhood of $\cdot 610\mu$ at low-level stations is clearly marked, and the observations show that the curves may be represented by broken straight lines. We cannot suppose, however, that the discontinuity at $\lambda = \cdot 610\mu$ will in reality be as sharp as the intersection of two straight lines would require. The precise cause of this well-marked point of discontinuity furnishes a point which must be left to further investigation.

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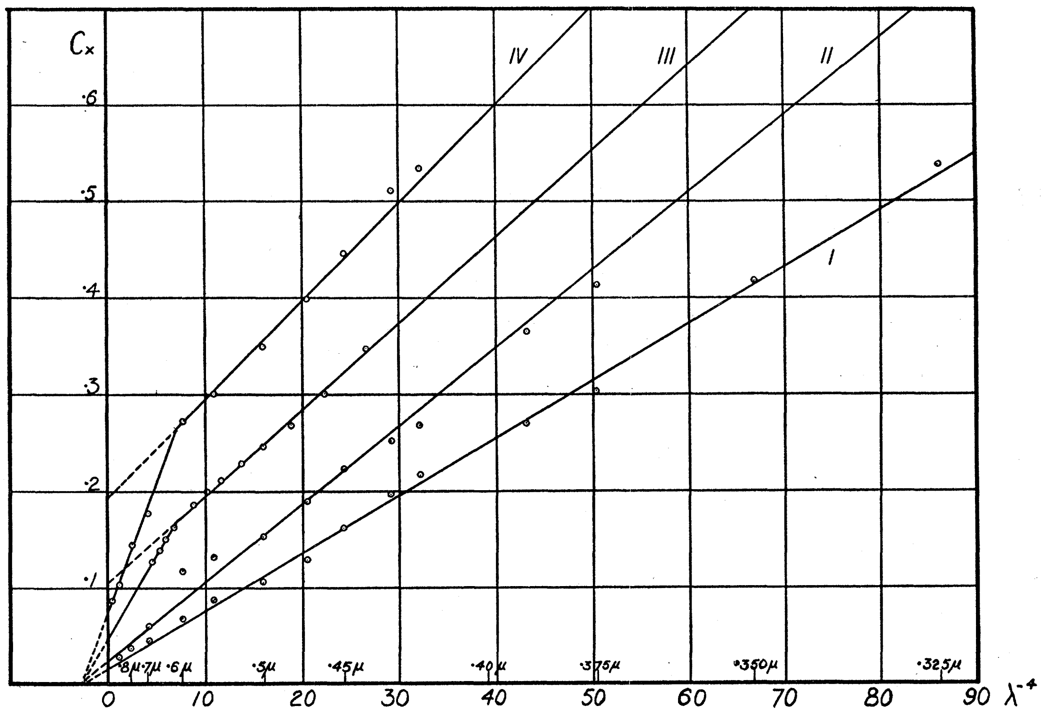

Diagram I. Variation of coefficients of atmospheric attenuation with wave-length.

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DIAGRAMS II.–VII.—*Relative Intensities from Zenith Sky Calculated for Different Wave-Lengths at Mount Wilson from Mean Coefficients of Attenuation.*

The curves given on Diagrams II.–VII. are calculated for various wave-lengths from formula (64) in terms of the mean coefficients of attenuation for Mount Wilson given in Table II. The numerical values from which these curves are drawn are given in Table VI. The three outer curves of each set represent the extreme and mean solutions of the integral equation (64),

$$T(0, \zeta) = (4\pi)^{-1}[\tfrac{3}{4}(1+\cos^2\zeta)]\,S[P(0, \zeta)+Q(0, \zeta)].c/C,$$

where

$$P(0, \zeta) = Ce^{-C}\,G[C(\sec\zeta - 1)]$$

and

$$Q(0, \zeta) = \tfrac{1}{2}\frac{c}{C}\begin{pmatrix}\epsilon_2\\ \epsilon\\ \epsilon_1\end{pmatrix}\Phi(C, 0).$$

The inmost curve of each set gives the intensity from zenith sky neglecting self-illumination, *i.e.*, it represents the value of

$$(4\pi)^{-1}[\tfrac{3}{4}(1+\cos^2\zeta)]\,SP(0, \zeta).c/C.$$

In this way the contribution of self-illumination to the intensity of sky radiation is made clear. We notice the divergence of the extreme solutions for large values of the sun's zenith distance.

The curves just described resemble the curves obtained by NICOLS* in his observations on the intensities in the spectrum of zenith sky compared with corresponding intensities in the spectrum of an acetylene flame. The results given by NICOLS were taken at various times of the day (July 18, 1907) at Sterzing in the Tyrol (lat. 46° 54′ N., long. 11° 25′ E., elevation 3110 feet above sea-level). Numerous observations have been taken by CROVA,† ZETTWUCH‡, MAJORANA,§ and others.|| The results are not strictly comparable with theory since the attenuation coefficients at the time of observation are not given. The forthcoming observations of the Smithsonian Astrophysical Observatory on sky radiation at Mount Whitney¶ (August, 1910) will enable an accurate comparison to be made with the results calculated from the attenuation coefficients measured on the same day.

* NICOLS, E. L., "Theories of the Colour of the Sky," 'Physical Review,' XXVI., June, 1908, p. 497. An extensive bibliography on sky observations is given in this paper.

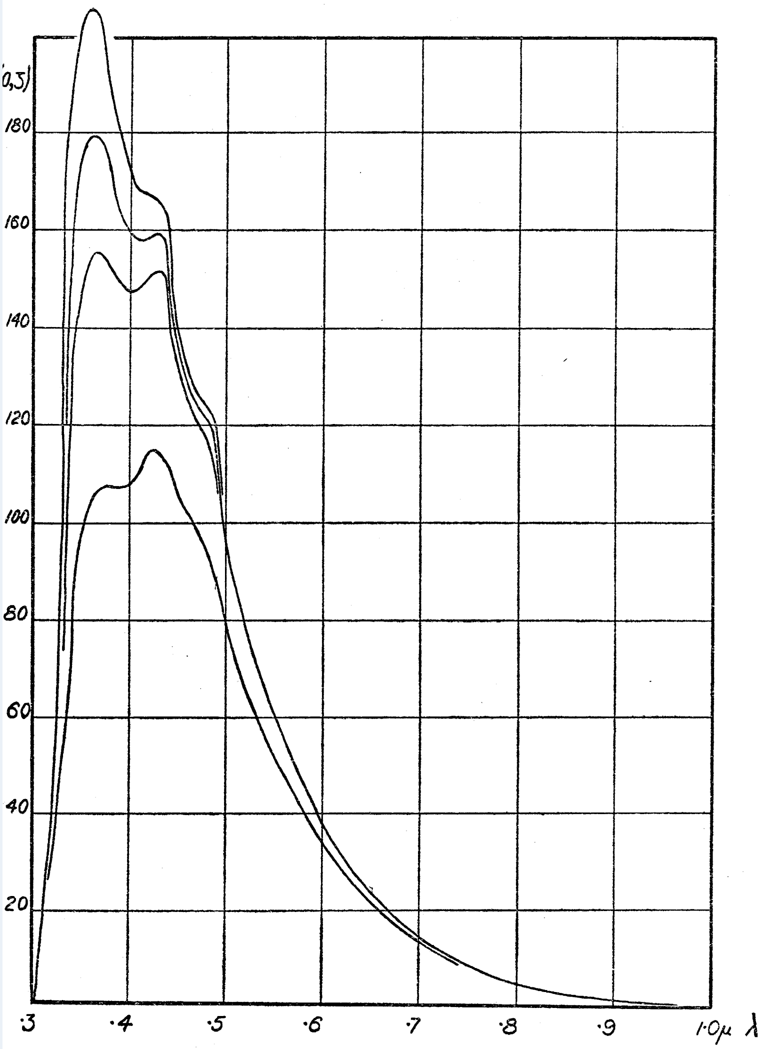
† CROVA, C. R., 'Comptes Rendus,' CIX., p. 493; CXII., p. 1178; also 'Annales de Chimie et de Physique,' (6), XX., p. 480.

‡ ZETTWUCH, G., "Ricerche sul Bleu del Cielo," translated in the 'Phil. Mag.,' (6), IV., August, 1902, p. 199.

§ MAJORANA, Q., "On the Relative Luminous Intensities of Sun and Sky," 'Phil. Mag.,' (6), I., May, 1901, p. 555.

[|| The writer is indebted to Dr. OTTO KLOTZ, of the Dominion Observatory, Ottawa, for calling his attention to the work of EXNER along these lines. EXNER, 'Sitzungsbericht d. K. Akad. d. Wissen., Wien, M. N. Klasse,' 1909, vol. 118, IIA. A summary of EXNER's work and of the observations of WIESNER and SCHRAMM, W., is given by ABBOT, C. G., in his recent book, 'The Sun,' p. 299 (Appleton and Co., 1911). A problem of somewhat the same nature as the present one is considered by GOLD, E., "The Isothermal Layer of the Atmosphere and Atmospheric Radiation," 'Roy. Soc. Proc.,' A, vol. 82, 1909.—*Note added December* 31, 1912.]

¶ ABBOT, C. G., "Report on the Astrophysical Observatory," 'Annual Report of the Smithsonian Institution,' 1911, p. 65.



gram II. Relative intensities from zenith sky calculated from Mount Wilson observations. $\zeta = 0°$.

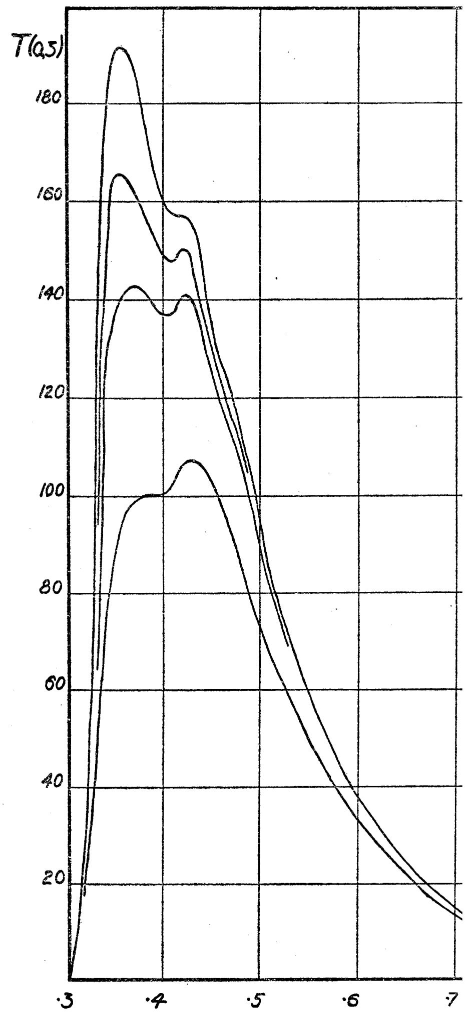


Diagram III. Relative intensities from ze
Wilson observations.

(*To face p.* 426.)

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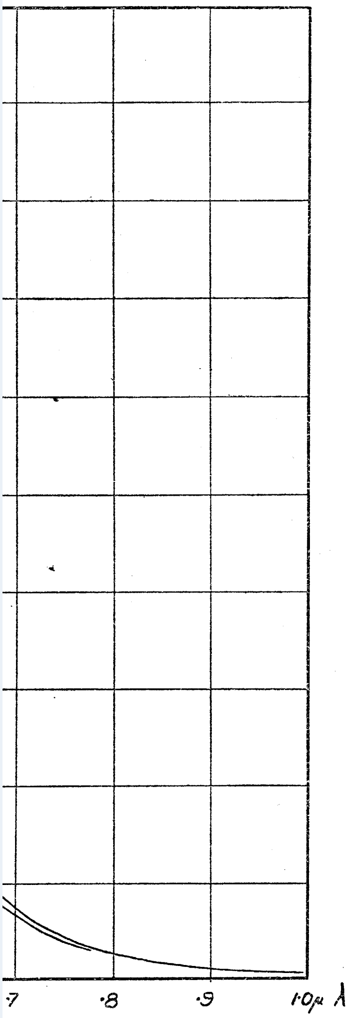

zenith sky calculated from Mount
s. ζ = 20°.

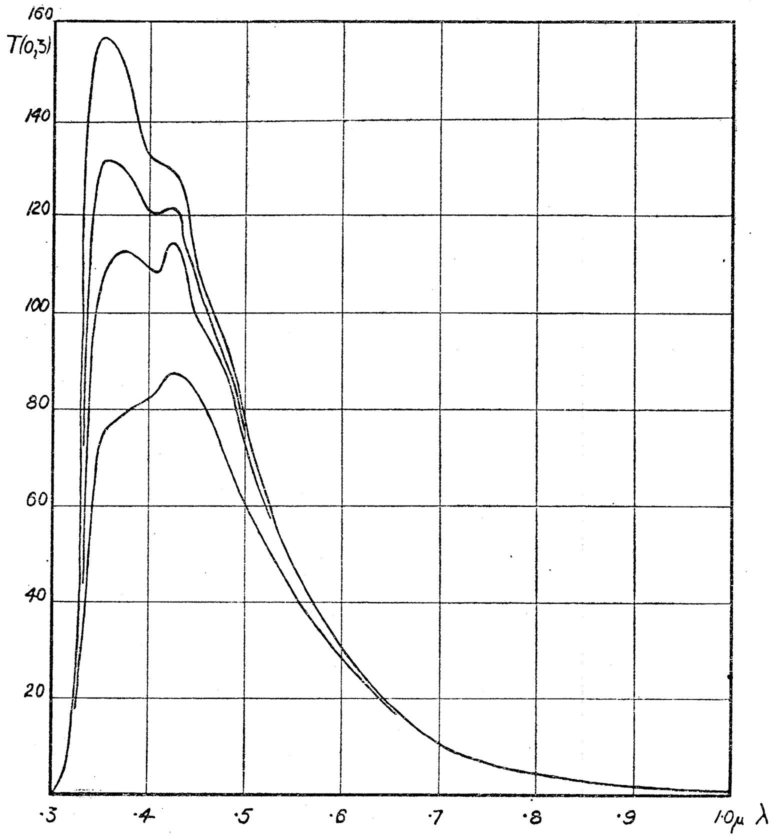

Diagram IV. Relative intensities from zenith sky calculated from Mount Wilson observations. ζ = 40°.

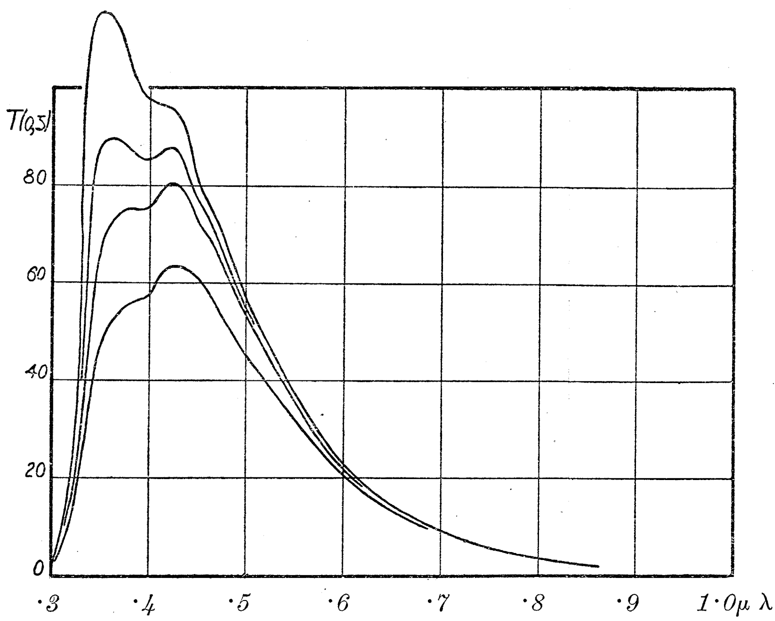

Diagram V. Relative intensities from zenith sky calculated from Mount Wilson observations. ζ = 60°.

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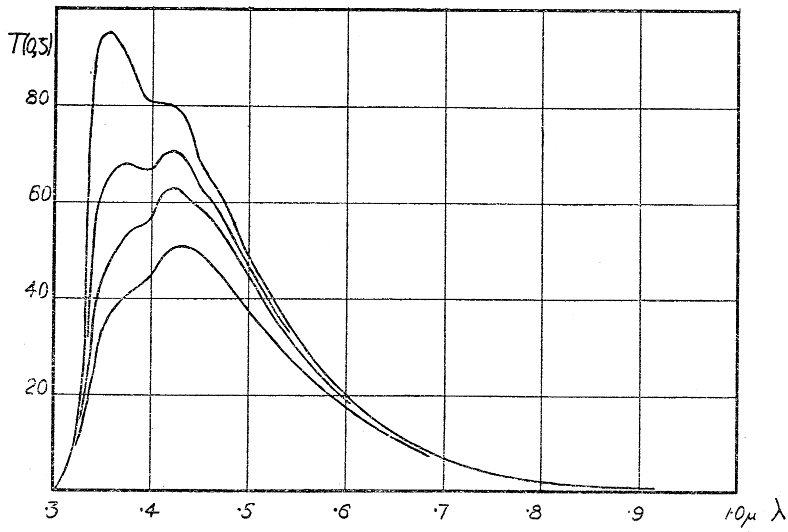

Diagram VI. Relative intensities from zenith sky calculated from Mount Wilson observations. $\zeta = 70°$.

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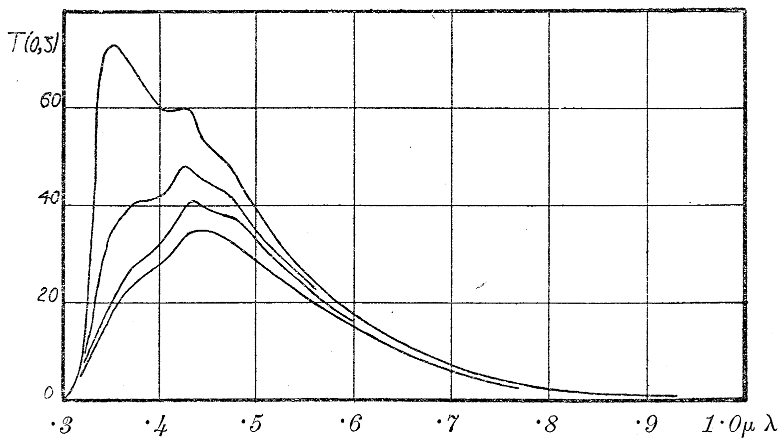

Diagram VII. Relative intensities from zenith sky calculated from Mount Wilson observations. $\zeta = 80°$.

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DIAGRAM VIII.—*Comparison of Formulæ for the Relative Intensities of Sky Radiation Calculated from Mount Wilson Observations for* $\zeta = 60°$.

It is of some interest to compare with the formula for sky intensity derived in the present paper the various formulæ which have been employed by various writers in the interpretation of sky observations. The results are shown graphically in Diagram VIII. for a zenith distance of the sun of 60°, and refer to intensities of zenith sky.

Curve I. is given by the formula (63)

$$(4\pi)^{-1} S \tfrac{3}{4} (1 + \cos^2 \zeta) C,$$

and is the same as that given by KELVIN* and employed by PERRIN† in an estimate of the number of molecules per unit volume of a gas.

Curve II. is given by the formula

$$(4\pi)^{-1} S \{\tfrac{3}{4} (1 + \cos^2 \zeta)\} c,$$

and is seen to give a considerably smaller value of sky intensity in the red, agreeing closely with the value $T(0, \zeta)$ for wave-lengths greater than $\cdot 45\mu$.

Curve III. is given by the mean solution (64) for $T(0, \zeta)$ which takes into account the effect of self-illumination.

Curve IV. gives the intensity of zenith sky neglecting self-illumination but taking into consideration the attenuation of both the incident and scattered radiation.

* KELVIN, 'Baltimore Lectures' (1904), p. 313.

† PERRIN, J., 'Annales de Chimie et de Physique,' 8me Série, September, 1909. See p. 79 of 'Brownian Movement and Molecular Reality' (F. SODDY, 1910), published by Taylor and Francis, London.

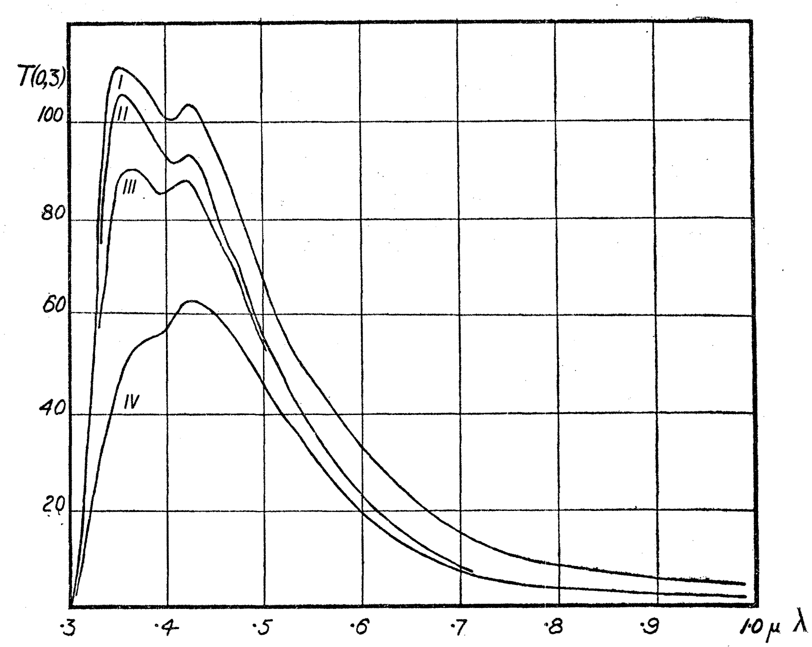

Diagram VIII. Comparison of formulæ for intensities from zenith sky, Mount Wilson, $\zeta = 60°$.

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DIAGRAMS IX. AND X.—*Total Radiation Calculated from Mount Wilson and Washington Observations.*

The curves drawn on Diagrams IX. and X. are drawn from the data calculated in Tables VII. and IX. for the values of total solar and sky radiation at Mount Wilson and Washington respectively. The various curves denoted by I., II., III., IV., refer to the following quantities:—

Curve I. gives the total intensity of solar radiation in calories per square centimetre per minute normal to the sun's rays; *i.e.*, the curve represents the variation of

$$\int_0^\infty \mathrm{E}(\zeta)\, d\lambda \text{ with } \zeta.$$

Curve II. gives the total intensity of solar radiation in calories per square centimetre per minute on a horizontal plane; *i.e.*, the curve represents the variation of

$$\cos\zeta \int_0^\infty \mathrm{E}(\zeta)\, d\lambda \text{ with } \zeta.$$

Curve III. represents the total intensity in calories per square centimetre per minute of the radiation from the entire sky on a horizontal plane; *i.e.*, the curve represents the variation of

$$\int_0^\infty \mathrm{H}(\zeta)\, d\lambda \text{ with } \zeta.$$

Curve IV. represents the total radiation from sun and sky on a horizontal plane; *i.e.*, the curve represents the variation of

$$\int_0^\infty \{\cos\zeta\,\mathrm{E}(\zeta) + \mathrm{H}(\zeta)\}\, d\lambda \text{ with } \zeta.$$

The calculations are only given as far as $\zeta = 80°$. Curves I. and II. fall to zero at $\zeta = 90°$. Curve III. is arbitrarily carried on beyond this point (in dotted lines) to $\zeta = 100°$, and represents to some extent the intensity of scattered radiation due to the "twilight" following the sunset or preceding the sunrise. The effect of the earth's curvature makes it practically impossible to calculate the scattered radiation when the sun is below the horizon. The heat reaching the earth before sunrise shows itself in continuous air-temperature records in the fact that the minimum in the diurnal inequality of temperature occurs about one-half hour before the sun actually rises.

The curve just described are intended for use in meteorological problems.

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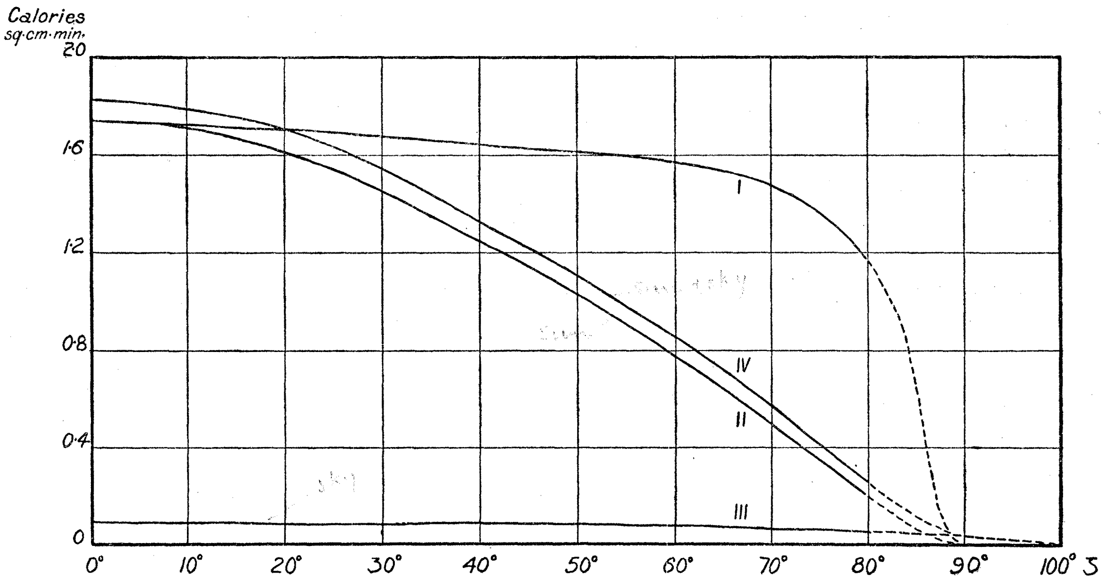

Diagram IX. Total solar and sky radiation calculated from Mount Wilson observations.

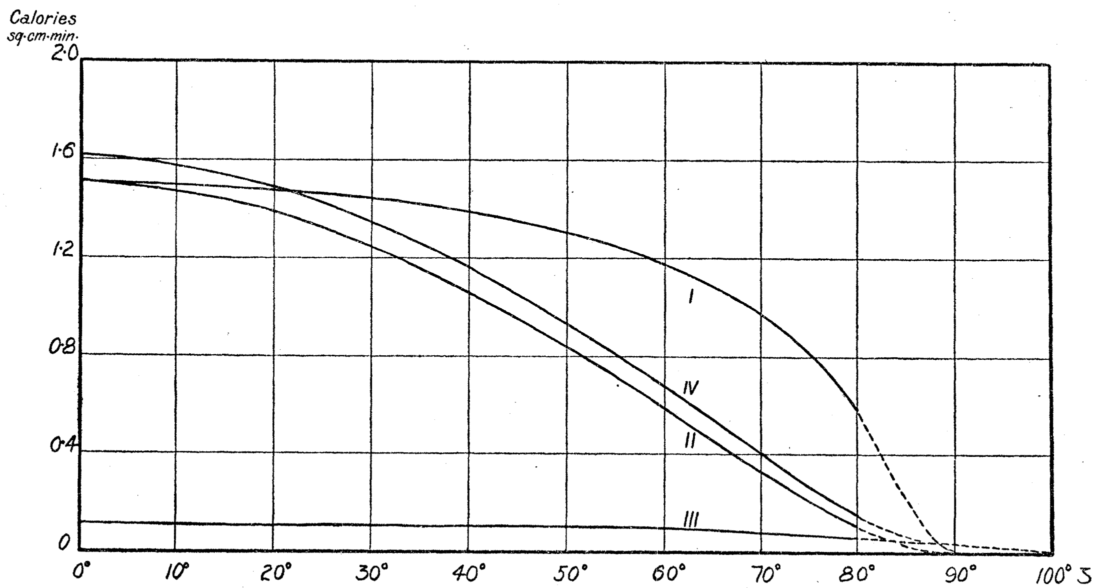

Diagram X. Total solar and sky radiation calculated from Washington observations.

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DIAGRAM XI.—*Relative Intensities from Sky and Sun Calculated from Mount Wilson and Washington Observations.*

The curves given in Diagram XI. represent the ratio *zenith sky/sun* for equal solid angles, the total radiation covering all wave-lengths being taken in both cases. The curve represents the formula

$$\omega \int_0^\infty \mathrm{T}(0, \zeta)\, d\lambda \Big/ \int_0^\infty \mathrm{E}(\zeta)\, d\lambda,$$

where ω is the solid angle subtended by the sun, *i.e.*, $\omega = 2\pi(1 - \cos 16') = 6 \cdot 80 \times 10^{-5}$.

Curves I. refer to Mount Wilson and include the extreme and mean solutions in the expression for $\mathrm{T}(0, \zeta)$.

Curve II. refers to Washington and represents the mean solution of the integral equation only.

Since the ratio just described is that most easily obtained in measurements of sky radiation, the curves just given are appropriate for a comparison of theory with observation.

DIAGRAM XII.—*Relative Intensities from Zenith Sky, Calculated for Different Wave-Lengths from Washington Observations.*

The curves given in Diagram XII. are drawn on the same plan as those described in Diagrams II.–VII. for Mount Wilson. In the present instance the numerical values from which the curves are drawn are given in Table VIII. Intensities corresponding to the mean solution of the integral equation are alone given, since the presence of "dust" at low-level stations is an extremely variable factor. This factor gives rise to a sharp bend in the attenuation curve for Washington shown in Diagram I. in the neighbourhood of $\cdot 610\mu$. The effect on the intensity of zenith sky is very sharply marked and gives rise to a finite discontinuity in the intensity curve corresponding to that wave-length. In reality the bend of the attenuation curve in Diagram I. cannot be as sharp as the intersection of two straight lines, so that the discontinuity in the sky intensity curve in this neighbourhood is probably represented by a sharp peak (represented in the drawing by the dotted portions of the curves) which arises from the effect of "dust" in scattering the long-wave radiation. The prominence of this peak in the intensity curve must be an extremely variable factor, and may possibly account for discrepancies in observations on sky-radiation; *e.g.*, NICOLS* mentions as a result of his measurements that sky intensities show far greater relative intensities in the longer wave-lengths than one would expect from RAYLEIGH'S theory. The effect of "dust" is an illustration of the effect of several groups of molecular complexes or small particles on radiation travelling through a medium containing them; each group which is able to produce a sharp bend in the attenuation-curve at any wave-length is also able to produce a corresponding maximum in the intensity-curve of the scattered radiation, the whole effect resembling a broad emission band in its spectrum.†

* NICOLS, *loc. cit.*, p. 502.

[† Important experiments describing the transition from lateral scattering to selective reflection are described by WOOD, R. W., under the name "resonance radiation." These are described in the 'Clark University Lectures,' Clark University, 1912, p. 101.—*Note added December* 31, 1912.]

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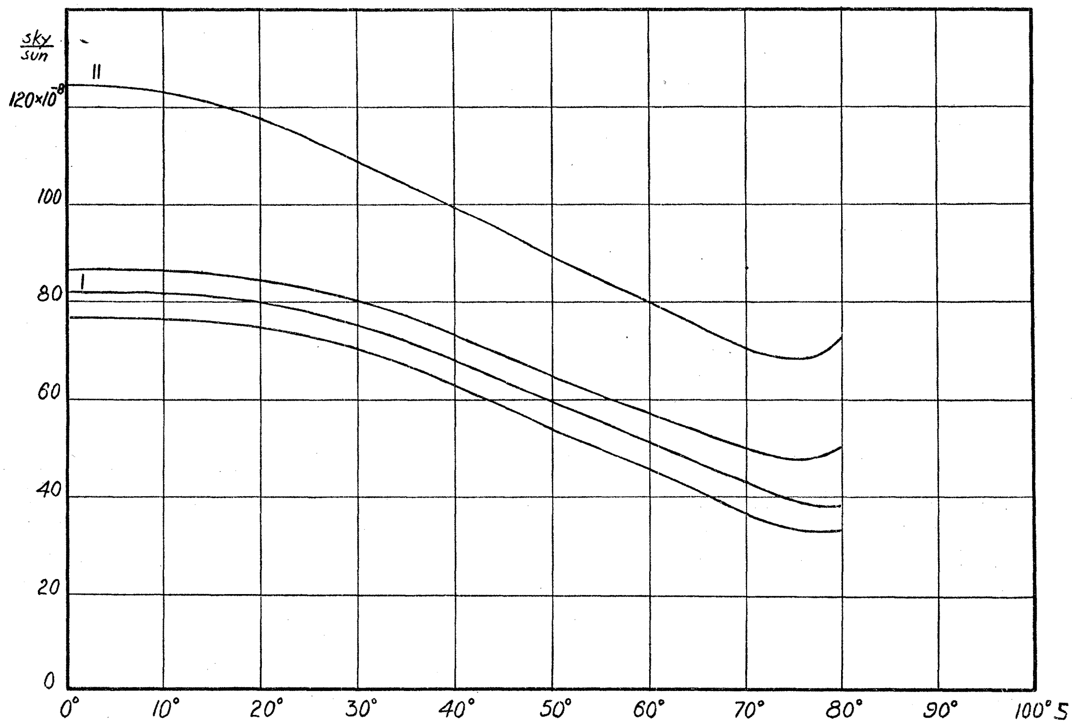

Diagram XI.—Ratio *zenith sky/sun* for equal solid angles calculated from Mount Wilson and Washington observations.

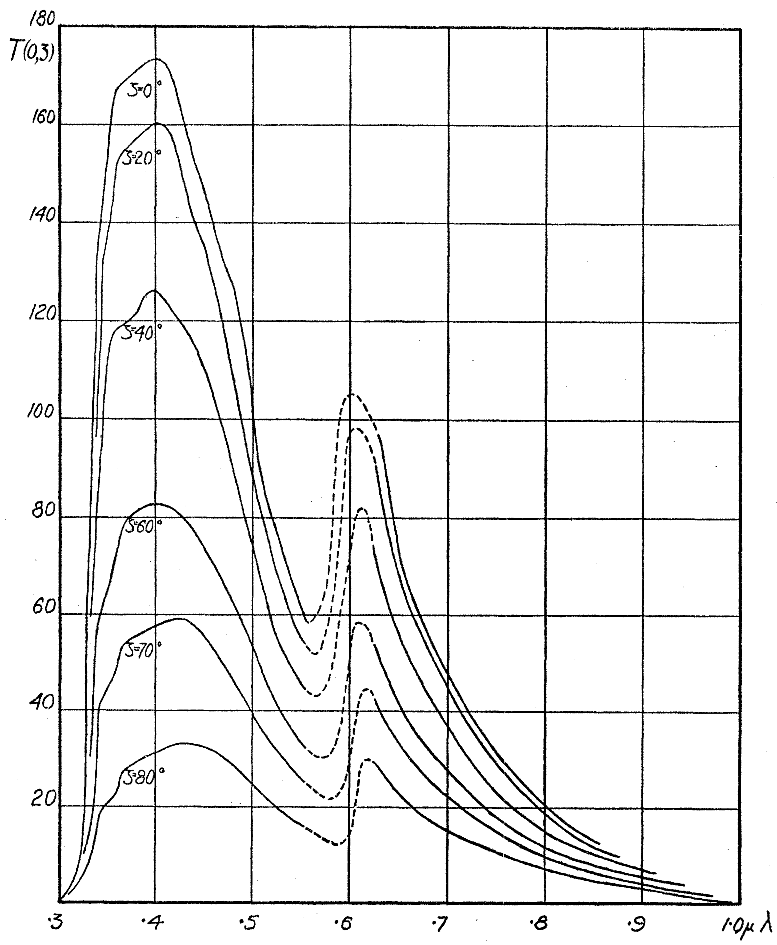

Diagram XII. Relative intensities from zenith sky, calculated from Washington observations.

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DIAGRAM XIII.—*Quality of Sky Radiation Calculated at Mount Wilson and Compared with Observations.*

The curves (i), (ii), (iii) of Diagram XIII. are plotted from the results of Table X., and show graphically how observed values of sky radiation agree with values calculated from mean attenuation coefficients at Mount Wilson. The curves corresponding to the extreme and mean values of the solution of the integral equation are retained throughout. Observed points are denoted by small circles, and lie fairly close to the calculated curves.

DIAGRAM XIV.—*Polarization of Zenith Sky Calculated from Washington and Mount Wilson Observations.*

The curves given in Diagram XIV. are drawn from data calculated in Table XIII., and represent the ratio of the component intensities of zenith sky polarized in the principal plane to the component polarized at right angles to that plane. This ratio is represented roughly by the formula

$$\frac{T_1(0, \zeta)}{T_2(0, \zeta)} = \frac{1 + Q(0, \zeta)/P(0, \zeta)}{\cos^2 \zeta + Q(0, \zeta)/P(0, \zeta)}.$$

Curves are given for various zenith distances of the sun, making use of the constants for Mount Wilson and Washington. In the latter case the effect of "dust" gives rise to a discontinuity in the neighbourhood of $\cdot 610\mu$, represented in an actual case by a peak in the curves (represented by dotted lines). The characteristic just mentioned is actually observed in curves of polarization of zenith sky obtained by NICOLS.* In comparatively "dust-free" air as at Mount Wilson this peak is wanting, and the curve represents a type obtained by NICOLS† which rises with increasing wave-length. Since the forms of the curves just described depend essentially on the effects of self-illumination, these will be extremely sensitive to the presence of "dust," and will therefore vary greatly with time and place. Reflection of solar radiation from the earth's surface (*e.g.*, a snow-covered landscape) will affect in a marked manner the polarization of sky radiation independently of effects of self-illumination, and may therefore account for the great variety of results obtained by observation on this point.

* NICOLS, *loc. cit.*, p. 508, fig. 10, Curves (*b*) and (*b'*).

† NICOLS, *loc. cit.*, fig. 10, Curve (*c*).

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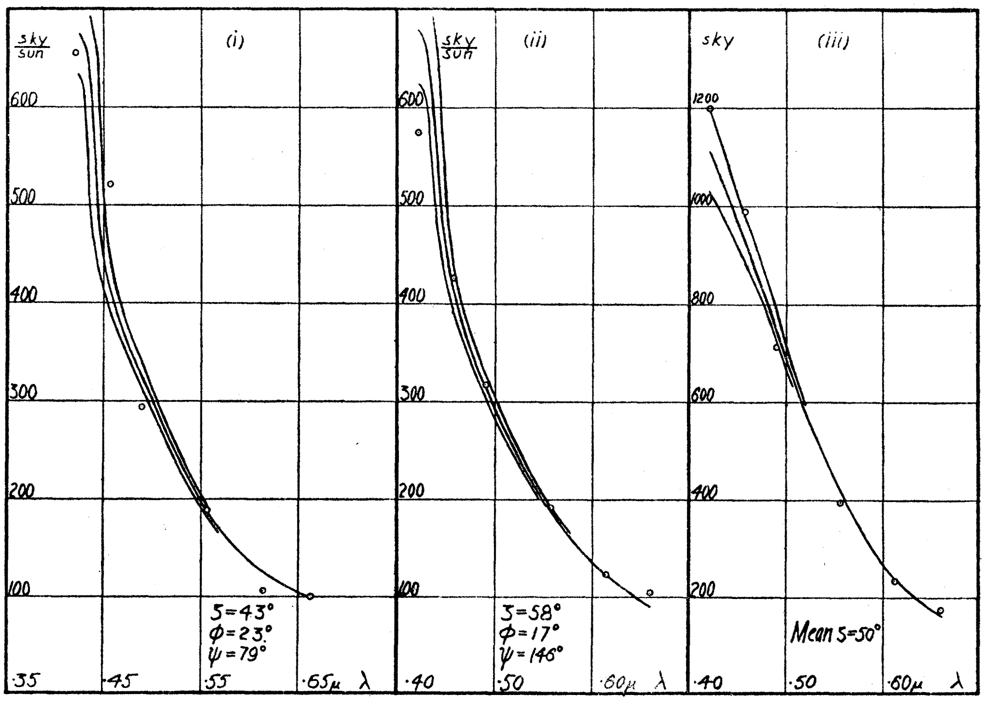

Diagram XIII. Quality of sky radiation, calculated from Mount Wilson observations, October 17, 1906.

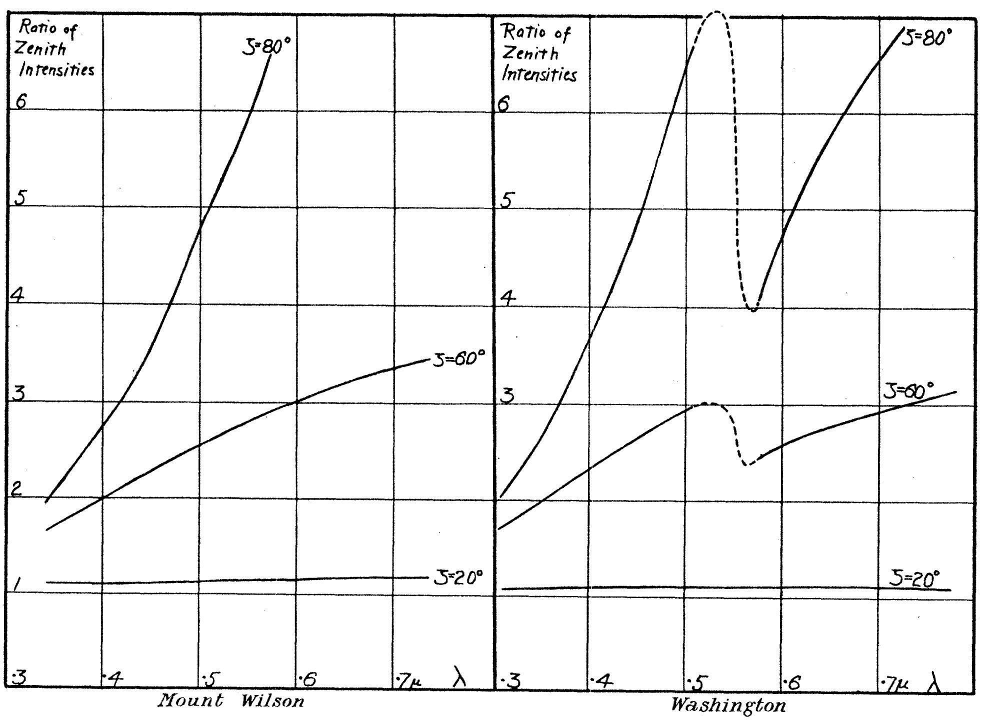

Diagram XIV. Polarization of zenith sky calculated from Mount Wilson and Washington observations.

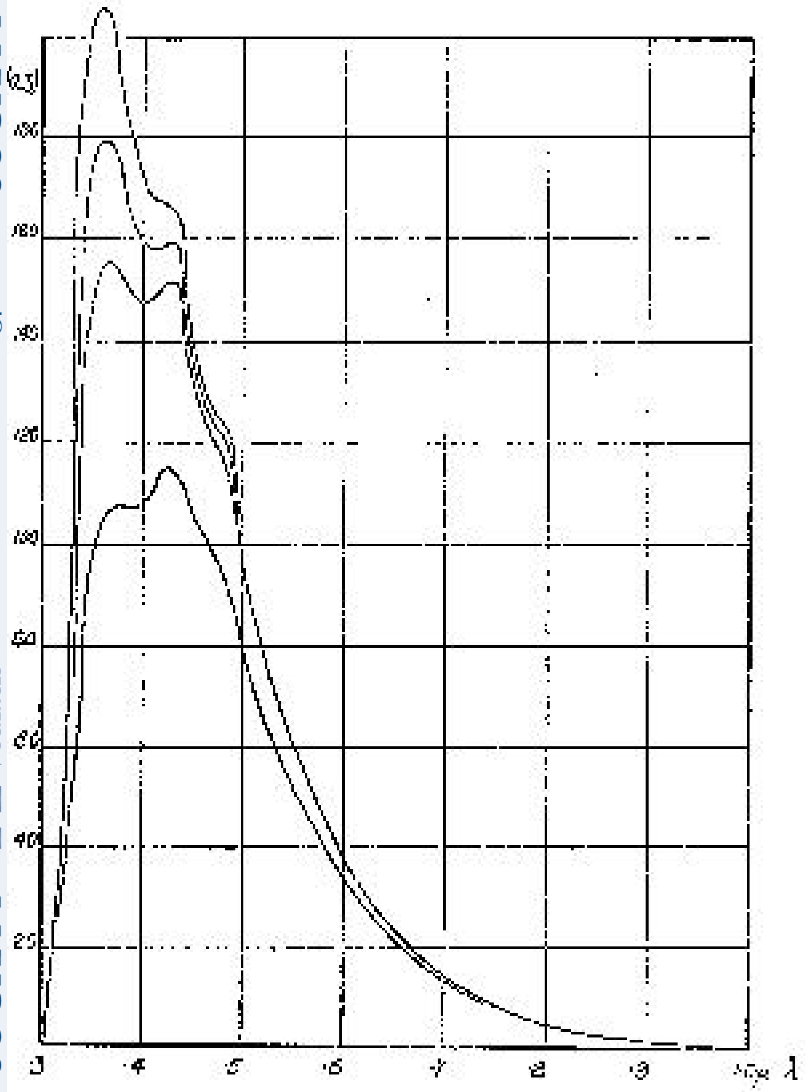

Diagram II. Relative intensities from zenith sky calculated from Mount Wilson observations. ζ = 0°.

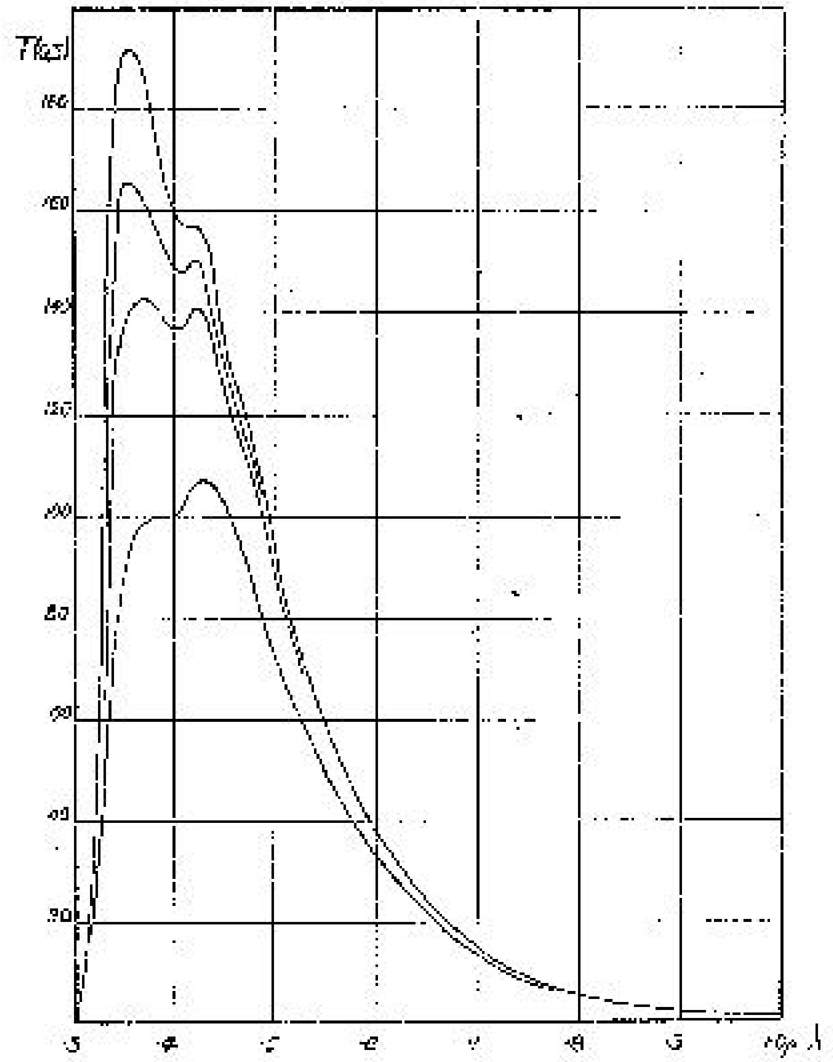

Diagram III. Relative intensities from zenith sky calculated from Mount Wilson observations. ζ = 20°.

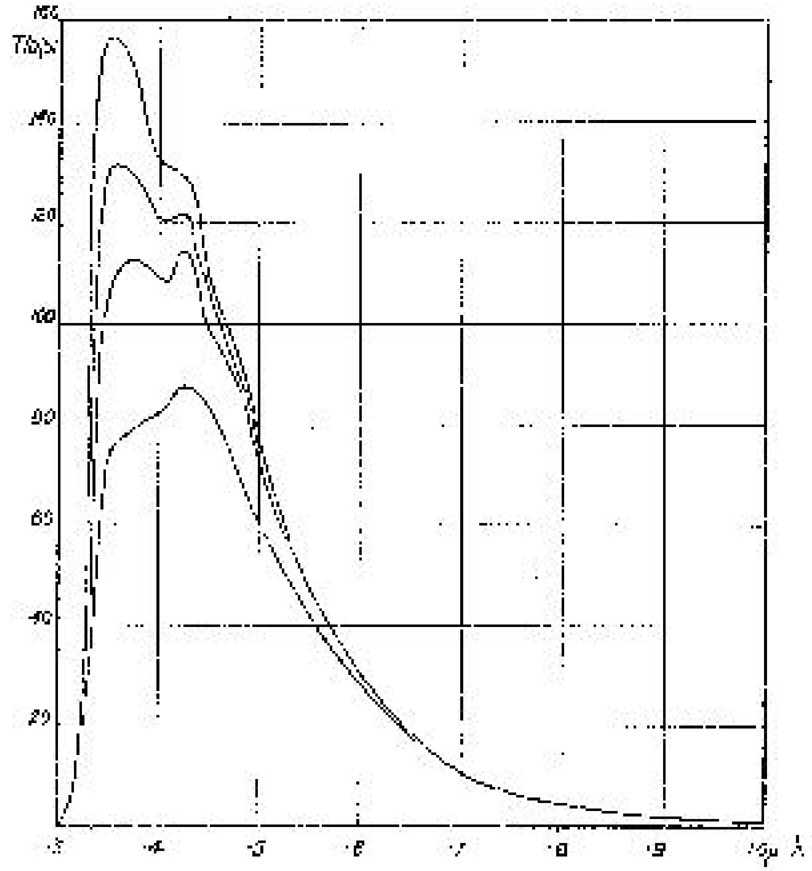

Diagram IV. Relative intensities from zenith sky calculated from Mount Wilson observations. ζ = 40°.

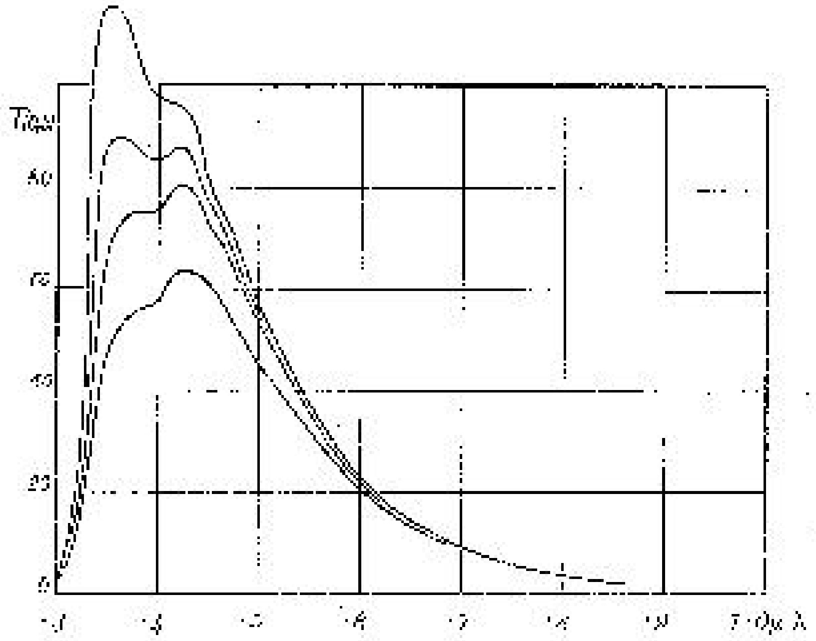

Diagram V. Relative intensities from zenith sky calculated from Mount Wilson observations. ζ = 60°.

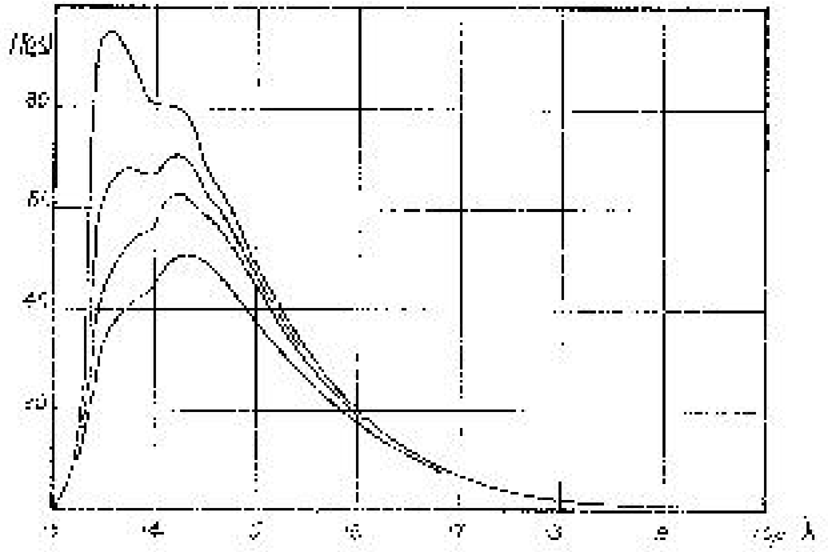

Diagram VI. Relative intensities from zenith sky calculated from Mount Wilson observations. ζ = 70°.

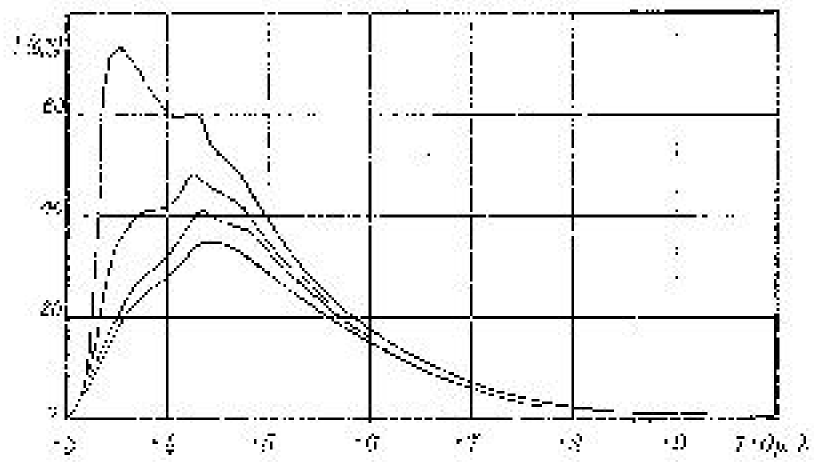

Diagram VII. Relative intensities from zenith sky calculated from Mount Wilson observations. ζ = 80°.